# MATH ALIVE!

Shashidhar Jagadeeshan

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# PREFACE

This resource book is based on the Mathematics Mela, an effort at the Centre For Learning (CFL) in Bangalore. True to the spirit of a “mela”, the event and the work leading to it were nothing less than a joyful joint celebration of mathematics by the students and teachers. Preparations started in late 1995 and intensified in the weeks immediately preceding March 23rd, 1996. On that day, many visitors, the present writer included, had the privilege of walking around the mela for a few hours, sampling and savouring its content and spirit. The content is well conveyed in the pages that follow, and some of the spirit seeps through via the coloured illustrations, which show not just the mathematical exhibits but the people, involved. One was struck by the total involvement of the whole school, some eighty odd souls. The idea that mathematics was just for some special students and specialised teachers was truly laid to rest. Bringing in nature, everyday experience, puzzles, art, tradition, and history ensured that there was something for everyone. Horizontal partitions between age groups and vertical ones between different parts of the subject were underplayed. At the same time, there was enough “advanced” material to challenge those with such inclinations. There was no doubt that the whole school both enjoyed and learnt from the experience.

One must ask how this resource book could be useful in more conventional settings. The spontaneity and gaiety of the mela should not blind us to the meticulous and detailed preparation which went into it and the reader will be guided to very stimulating and tested material by going through the book. This material can fit into the classroom, or a maths club setting, or, for the adventurous, a mela of one’s own. Not everyone may be able to assign a whole month to mathematics related activity. Even if one could, it would take a whole team to keep the students gainfully occupied. But the maximum time any teacher is likely to give to such an activity is small in comparison with long weeks, months, and years spent on the nitty gritty of school mathematics. For this small investment, the motivation and enthusiasm generated are already a rich return; with actual understanding or skill being a bonus so to speak. If every student can emerge from the school system knowing something of the treasures that lie below the dry landscape of the standard syllabus, surely some will go on to explore them further, and the enrichment of others is as important.

Many practitioners and teachers of mathematics do feel some unease at anything which might seem to compromise the rigour and structure of their discipline. They have a point if the target audience consisted solely of people like themselves. Indeed, those driven by talent and temperament to do mathematics seldom need much more than this same rigour and

structure to keep themselves deeply happy, the mathematics being its own reward. Even they could not lose by adopting as a temporary and willing suspension of disbelief, the more open ended approach of which this book is a sample. After all, it is a question of a month or two. But for the great silent- and sometimes perplexed – majority who must do other things, this month could make all the difference to their relationship with mathematics. To draw a parallel, music is an activity which, at the highest level, demands skill, training, talent, and rigour of a high order. But would music not be the poorer if there was no scope for exposure, enjoyment, appreciation, and some participation especially at the basic level? Perhaps it is no accident that CFL also throws open the gates of music to its members! In any case, this book from them urges us to change the way we look at, teach, and learn mathematics.

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# PROLOGUE

"*Many who have never had the occasion to discover more about mathematics confuse it with arithmetic and consider it a dry and arid science. In reality, however, it is a science which demands the greatest imagination*" - Sofia Kovalevskaia

Those of us who love mathematics and teach it, see that it is intrinsically beautiful and has great order. Why, then, are so many children in almost all cultures frightened or bored by mathematics? We as teachers seem to convey a sense of fear and helplessness with regard to mathematics instead of conveying its beauty.

**Plate 0. A CFL student during the Mathematics Mela**

Although students go through a fairly rigorous course in mathematics (often including calculus), they rarely come away with an understanding of what the nature of mathematics is or what mathematicians really do. Mathematics is thought of as a theoretical subject to be done only with pencil and paper, with unfortunately little room for experimentation or hands-on work. This attitude has defined how mathematics is taught.

For the most part mathematics has been taught using a black board; concepts are presented, followed by a series of problems to test the understanding of these concepts. Most topics covered involve the use of formulae and demand a lot of computation, rather than giving the students a feel for the way modern mathematics is done, with its emphasis on proofs. Topics are introduced and taught in a linear fashion with no room for exploring by-lanes or alleys. Although a concept might have taken years to develop, with many false starts and mistakes, it is presented to the student in its final and most polished form. While this is

perhaps necessary for clarity of exposition, the student remains unaware of the historical evolution of the ideas, and the human drama behind the topics he or she has studied. Mathematics traditionally taught (at all levels) seems to give the impression that there is only one way to solve a given problem. There is really no room for discussion, where the student can contribute at a level comparable to the teacher. Since all information flows from the teacher, students immediately set him or her up as an authority. This feeling is further strengthened by the fact that students are rarely exposed to open-ended problems, i.e., problems to which the teacher does not already know the solution. All this can lead to a sense of insecurity in many children regarding mathematics.

There is further a feeling among students that mathematics does not lend itself to experimentation; that there is only one given way to approach a problem. They do not feel free to play around with ideas and perhaps discover their own method. There is also the feeling that mathematics is to be done either mentally or using pencil and paper. However, many great mathematicians like Archimedes actually used hands-on methods like building models etc., to understand and solve problems in unusual and creative ways.

At Centre For Learning we have addressed many of the issues just raised. Our curriculum is not dictated by examinations, and this makes the learning of mathematics (indeed all subjects) free of the pressure and fear that competitive evaluation brings. This has helped the children to enjoy mathematics for its own sake. Moreover, our education stresses the need for a healthy relationship between the teacher and student, and this allows the student to relate to the teacher without a sense of authority or fear.

For children under 10 years, mathematics is learned almost completely through activities. For example, to teach the concept of 'units of measurement', they begin by measuring lengths in hand spans, weights using stones and so on. When they have a concrete feel for what a 'unit' is, they are ready to use any of the standard units of measurement. For the older children too wherever possible we have used activities to introduce or reinforce the understanding of a concept in mathematics. To illustrate: while teaching the notion of loci the following project was given. Take a cardboard and affix a string (not taut) between two thumbtacks. Using a pencil along the string trace the curve you obtain. Can you describe the

figure you get? What is this figure a locus of? What happens if the thumbtacks are brought closer and closer together?

To give children a sense that not all questions in mathematics have clear-cut solutions, they are exposed to open problems, which they can actually comprehend. They are set small research problems for which perhaps even the teacher may not have a solution.

In the academic year 1995-1996, we chose the theme of Mathematics for our annual Mela (carnival). Throughout the year, we engaged in activities and covered topics that gave children a spatial sense, a feel for recognizing patterns, recognizing mathematics in nature and perhaps even a feel for what it means to investigate an open problem. We felt these would give the students a far better sense of the nature and beauty of mathematics than would the usual school syllabus.

Math Alive! is an outcome of our experience with the mathematics mela. Based on an in-house report on the mela it is meant to serve as a resource book for teachers of mathematics. The object of the book is precisely what the title suggests – to make mathematics come alive in the life of a child. The teachers if they wish can create their own mela (see Chapter VIII on how this can be done) using topics suggested, or use individual topics to enrich their mathematics curriculum. It is not necessary that a teacher repeat the projects verbatim. What is intended is that the document be self sufficient, in the sense that it gives enough information, ideas and suggestions for the teacher to go ahead and do the projects described. Using the references mentioned and other sources each teacher is free to create her or his own version of the projects chosen.

It has been a very interesting experience compiling this resource book. We sincerely hope that our readers enjoy it and find it useful. We welcome suggestions and hope that this book may stimulate a dialogue on the teaching of mathematics.

It is the onus of mathematics educators to create a curriculum which helps children enjoy mathematics and see it as something beautiful to be done. I only hope that this document serves as a starting point for such an attempt.

# I

# TESSELLATION - THE ART OF TILING

*A mathematician, like a painter or a poet, is a maker of patterns....the mathematician's patterns, like the painter's or the poet's, must be beautiful.* - G.H. Hardy

## Introduction

The dictionary meaning of the word tessellate is to 'pave with tesserae' or 'fit together exactly' and tesserae in Latin means 'one of the small pieces of which a mosaic is made'. A synonym for tessellation is tiling, and a formal (and formidable!) definition of the tiling of a plane T is *a countable family of closed sets, which cover the plane without gaps or overlaps.* The basic idea is to find shapes that fit together in such a way that they completely cover a two-dimensional plane without any gaps.

Tiling has fascinated mankind for centuries and perhaps the art of tiling began as soon as people started building homes. Examples of tiling abound in all cultures, from the mausoleum of I'tamad ud-Dawla, at Agra, Alhambra of the Moorish builders to the beautiful quilts of the American Indians. All cultures have played with designs that fit together in repetitive ways, and yet yield fantastic patterns. In this chapter based on our experience of the Mathematics Mela at CFL, we give an outline of how a project on tessellations can be conducted. We also give a brief outline of the mathematics that can be taught using tessellations.

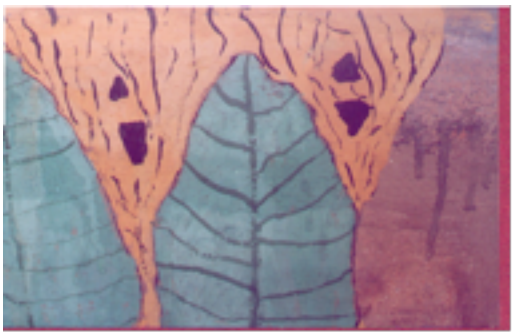


**Plate I.1 A tessellation mosaic created by students of CFL**

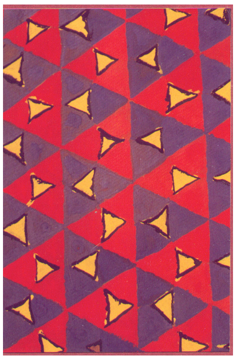
Tessellations offer a child the unique opportunity of combining geometry with art. Mathematicians have always been thrilled with the work of M.C. Escher who took the art of tiling to dizzying heights. During the mela we observed that the repetitive nature of tessellations, rather than boring the children, held their attention for long periods. Tessellations became a household word and children started observing tiling patterns all around them.

Children in the age group of 6-12 years are ideally suited for this project – given its tactile appeal. However you will find as we did that almost everybody in the school will be fascinated by tessellations, and will begin experimenting with their own. The older the age group, the deeper the mathematics that can be taught. The time one wants to spend on this project depends on the teacher and the curriculum requirements. The project has enough depth and potential for a group to spend 3 to 4 hours a week for a term (3 months), as we did.

**Tessellations as a Project**

Start the project by explaining what monohedral tiling (that is, tiling using only one shape) means by showing children samples of tiling and ask the question – which are the regular polygons that tile? It is a good idea to start with regular polygons, since children at this age are most comfortable playing with these shapes. You will discover that the only regular polygons that tile are – squares, triangles, and hexagons (why?). If necessary give children cut out shapes that they can colour and stick on a sheet of paper. Older children can cut out their own shapes after understanding how to construct regular polygons. Already you will find amazing artwork emerging from children.

**Plate I.2 Monohedral Tiling**



Now extend the idea of tiling using shapes other than regular polygons. You will discover that parallelograms also tessellate. How about circles? As one child put it, “A circle tessellates if you also use small diamond pieces”. Fair enough! In fact the next step is to allow more than one shape in a tiling pattern -–here the potential is endless.

How about irregular figures as found in the work of Escher? For this one needs the so-called ‘nibble’ technique (see [GC]). The altered shape (see Figure I.1), resulting from the ‘nibble’ technique must retain the same area as that of the original shape. The new shape is achieved by two possible transformations: the slide transformation and the rotation transformation.

The slide transformation (see Figure I.1): This transformation is restricted to polygons whose opposite sides are parallel and congruent, since an operation on one side affects the operation on the other. The square is the easiest to work with, and it is a good idea to start with it.

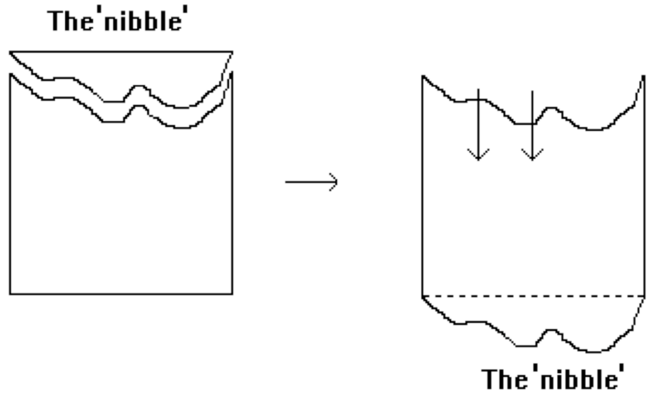



**Figure I.1.**

The 'nibble' technique is described below:

1) Colour one side of the square completely to avoid inadvertently flipping the piece while moving or taping the nibble.

2) Create a 'nibble' on one side of the square, starting at one of the corners and ending at the other. Note that the nibble should be cut carefully so that scraps or waste pieces are avoided.

3) The nibble is moved across to the congruent and parallel side.

4) Tape the nibble securely after matching the straight edges and corners correctly.

5) A second nibble can be cut from one of the other pairs of parallel sides and slid to the opposite side.

6) Tessellate by repeated tracing of a sample irregular tile without flipping or rotating while tracing.

The rotation transformation (see Figure I.2): This transformation is restricted to polygons with adjacent sides that are congruent.

Steps 1) and 2) are the same as above.

3) The nibble is rotated at one vertex onto the congruent adjacent side and taped.

4) A similar operation can be done with the other pair of adjacent sides.

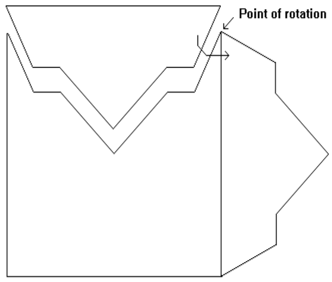



**Figure I.2.**

The idea can be extended in many ways. Mark the mid-point of each side (see Figure I.3). Now make the nibble from one corner to the mid-point that has been just marked. The nibble can be rotated about the mid-point and taped onto the remaining half of the side. Experiments can be made with beginning the cut at a corner and terminating it at any point before reaching the other corner. Another variation is to avoid corners at the beginning and end of the cuts on one side. This requires precise measurements to decide exactly where to affix the nibble on the opposite or adjacent side.

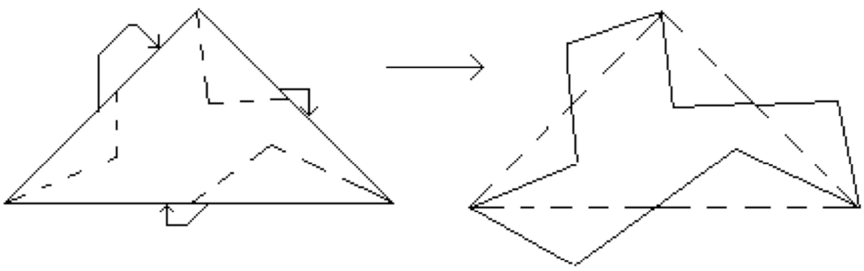


**Figure I.3**

Once children have learnt the nibble technique then you will see that they are ready to take off. Their art can be expressed in the form of murals, stained glass paintings, posters and in terracota (see Plates VIII.1, I.1, I.2, IV.1)

**The mathematics of tiling**

Even a cursory glance at the book by Grunbaum and Shephard ([GS]) suggests that the mathematics behind tiling is deep and is a rich area for research and investigation. According to Grunbaum and Shephard some of the most exciting developments in the area of tiling and patterns have occurred in the last thirty years or so.

At the school level we suggest that the following concepts in mathematics can be taught and reinforced using tessellations:

**Regular polygons and their tiling**

By the end of the project, children should be familiar with the regular polygons, their names and properties and how to construct them. Relevant to our project was the issue: which are the familiar geometric shapes that tile? That is, starting with one shape at a time, when three or more pieces come together, no gaps or overlaps should occur. Children have to figure out that regular polygons tessellate when the sum of the angles that surround a given vertex (see Figure I.4) is 360 degrees.

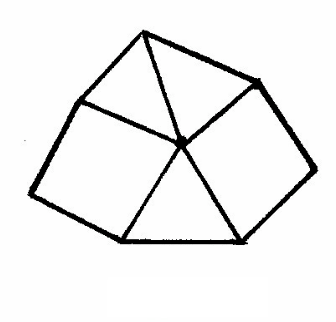


**Figure I.4.**

Tiling using only one shape is referred to as *monohedral tiling.* Many of the classical tilings like the one in Shibam-Kawkaban (Figures I.5a, I.5b), a minaret in Yemen (see [JW] for a detailed discussion), use more than one geometric figure, or are *polyhedral tilings*. It turns out that these figures can be encoded as follows: list the number of sides of the polygon, which surround any vertex of the tiling. For example Figure I.4 will receive the code (3,3,4,3,4) as a given vertex is surrounded by three triangles and two squares.

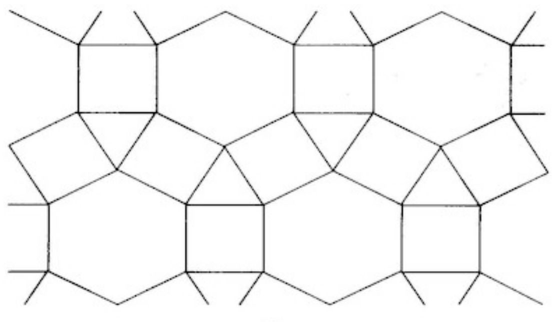



**Figure I.5. a. A tessellation of hexagons, squares and triangles, coded (6,4,3,4), from the Shibam-Kawkaban, a minaret in Yemen**. Reproduced from Julian Williams, *Geometry and Art*, 1993, Fig. 6.11a, Page 158,[JW]. © David Nelson, George Gheverghese, Joseph and Julian Williams, 1993. Reprinted **Multicultural Mathematics: Teaching Mathematics from a Global Perspective** by David Nelson, George Gheverghese, Joseph and Julian Williams (1993) by permission of Oxford University Press.

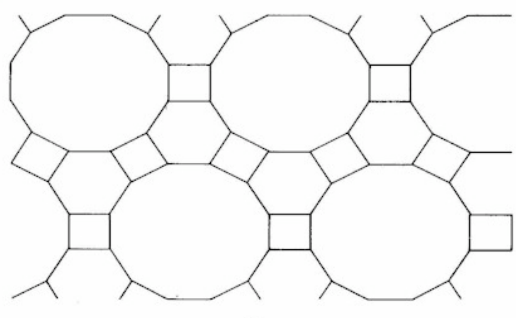



**Figure I.5. b. A tessellation of hexagons, dodecagons and squares, coded (6,12,4), from the Shibam-Kawkaban, a minaret in Yemen.** Reproduced from Julian Williams, *Geometry and Art*, 1993, Fig. 6.11b, Page 158,[JW]. © David Nelson, George Gheverghese, Joseph and Julian Williams, 1993. Reprinted **Multicultural Mathematics: Teaching Mathematics from a Global Perspective** by David Nelson, George Gheverghese, Joseph and Julian Williams (1993) by permission of Oxford University Press.

Extending the question about which polygons tessellate, one can ask - which are the codes that tessellate? For example (6,12,4) tessellates but (6,12,6) does not. Can one list all possible codes that tessellate? Again the answer to this question rests on the fact that the angles surrounding any given vertex must add up to 360 degrees. Given a code one can generate a tiling by choosing a vertex and drawing the polygons listed in the code around the given vertex. Now repeat this process by choosing a new vertex from one of the drawn polygons. One may or may not get nice patterns! Another interesting project could be, start with a famous tessellation from the past and ask students to come up with a code for it. If we do not restrict ourselves just to regular polygons the possibilities are immense (see [JW] and [GS]).

**Transformation and the nibble technique**

The nibble technique described above allows us to play with regular geometric shapes and change them into irregular shapes. Children can see that the area of the initial shape and final shape remain the same. Working on a hands-on activity like tessellation goes a long way in conveying to the child non-verbally, a concept that they may otherwise find difficult to understand. Tessellating is impossible if the area changes due to overlapping or loss of a piece while cutting.

One can introduce the operations of translation, rotation, reflection and glide reflection (reflection combined with translation). The nibble technique described above uses both the rotation and translation transformations. The use of these operations increases once again the possible figures that tessellate.

Illustrated below is the use of these operations to create a simple bird, and a stained glass window using the bird as a tile.

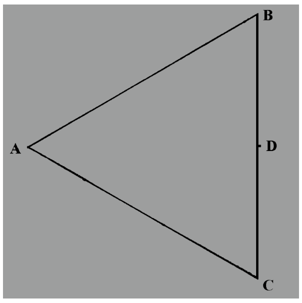



Figure 1.6 a

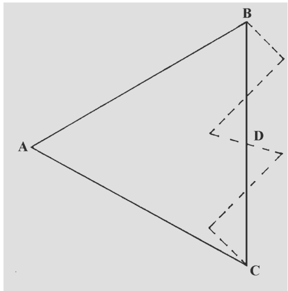



Figure 1.6 b

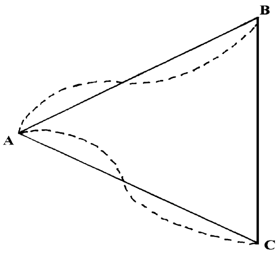



Figure 1. 6 c

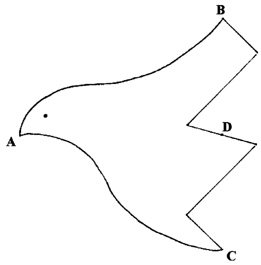



Figure 1.6 d

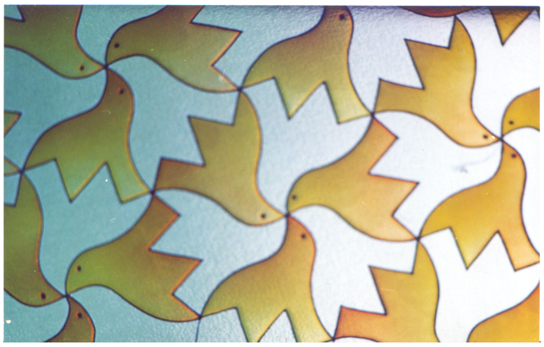


**Plate I.3 Stained glass window at Centre For Learning**

## Advanced Topics

Having introduced students to the various kinds of transformations (or rigid motions of a plane) it would be nice to get into symmetry. A **symmetry** of a figure is a rigid motion which leaves the figure unchanged. M.C.Escher the famous Dutch artist was a master at using symmetry to create amazing tessellations (see [LT] for a discussion of symmetry and tessellation).

If symmetry is involved can Group theory (see [MA], Chapter 5) be far behind? Crystallographers study the classification of patterns by their symmetric groups. The book by

Grunbaum and Shephard and some other recent books on tiling and algebra suggest that tiling could be an interesting starting point to study abstract algebra.

Finally the more ambitious could get into aperiodic and quasiperiodic tiling studying the work of Roger Penrose (who showed in 1974 how to create aperiodic tiling with just two different shapes) and others. According to [EL], "a set of shapes with the property that, though the whole Euclidean plane can be covered by non-overlapping replicas of shapes, no periodically repeating tiling pattern can be constructed for them".

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# II

# THE PLATONIC SOLIDS

*.... they symbolize in an unrivaled manner man's longing for harmony and order...*

- M.C. Escher

Platonic solids, also called the 'regular solids', are 3-dimensional geometric solids whose faces are all congruent regular polygons - like triangles or squares for example - and in which the same number of polygons meet at each vertex. Interestingly there are only five such Platonic solids: the tetrahedron, octahedron, icosahedron, cube and dodecahedron (see Figure II. 1).

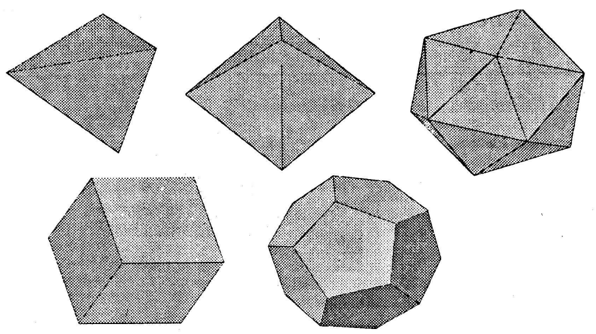

**Figure II.1: Tetrahedron, Octahedron, Icosahedron, Cube and Dodecahedron.**
Reproduced from Joseph Samuel, *Crossing Bridges*, Page 31, [JS]. © Resonance. Reprinted by permission of Resonance, Indian Academy of Sciences.

The history of these shapes is quite interesting (see for example, [HE]). The Greeks linked the five solids to the four basic elements (air, fire, water and earth) and made up for the difference by adding the heavens to the list of the elements! Specifically, fire was thought to take the shape of a tetrahedron, since it pointed upwards; the earth a cube, for it's stability; water an icosahedron, since it seems to 'roll' easily; and air an octahedron, for it's ability to spin when held between two fingers. To solve the problem of the remaining solid, the dodecahedron, which seemed to have no element to pair with, Plato decided that this was the shape "which the Gods used for arranging the constellations on the whole heaven." After all, there are twelve zodiac signs, and the dodecahedron has twelve faces! Even Kepler, in his earlier days, went to great lengths to justify the associations of planetary orbits to solids. In Kepler's time only six planets were known, and he managed to inscribe a regular solid inside each planets' "sphere" of orbit. Into Saturn's sphere he inscribed a cube, inside which another sphere was inscribed, the orbit of Jupiter. Inscribed in this sphere was a tetrahedron.

Next came the sphere of Mars, into which was inscribed a dodecahedron. This was followed by the Earth's sphere, into which fitted an icosahedron. Finally there was Venus' sphere, an octahedron, and inside it Mercury's sphere. Kepler even fitted the ratios of the orbits into his scheme. In his later life, when Kepler admitted that his initial enthusiasm had been misplaced, he still did not "regret any of the time spent and felt no fatigue. [He] was not afraid of cumbersome calculations...or whether [his] joy was to vanish into smoke."

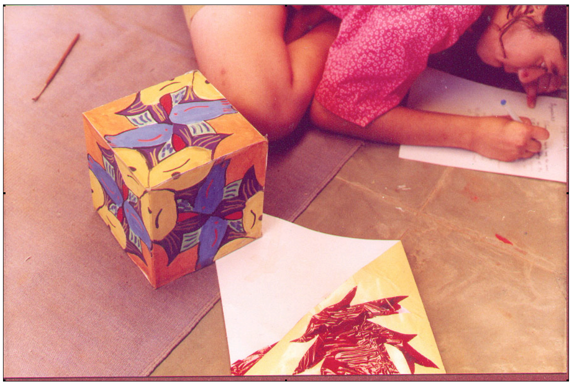

**Plate II.1 Tiling platonic solids**

**Platonic solids as a project**

A project involving the construction of the platonic solids and understanding the reasoning behind the proof that there can be only five such solids is ideal for the age range of 14-16 years. The project has enough depth and potential for a group to spend 3-4 hours a week for a term of 3 months, as we did. The payoff in terms of mathematics is tremendous. The process of constructing these solids reinforces several basic geometric concepts; students gain experience in spatial reasoning and begin to understand why only five such solids exist. Moreover they can be introduced to the idea of a proof, and how even though we may have sufficient evidence of a statement being true, there is still the need for a rigorous proof. Further, a hands-on familiarity with these solids will help them later on if they want to learn about their symmetry groups, etc.

Based on our experience at the Mathematics Mela, we recommend the following. Begin the project with a short history of the discovery of the five regular solids; work on producing the solids using cardboard and in the process come to a rudimentary reason for their being only five such solids. This will lead to a more rigorous proof and to an understanding of the idea of a 'proof' itself. First present the proof due to Euclid. Then use the method of solving a set of inequalities derived from Euler's theorem for solid figures. If time permits, study the properties of the five regular solids and work on related activities: painting, growing crystals, making soap films and so on.

**Construction and two proofs**

To investigate the shapes of the solids, start with making models. Draw nets (unfolded versions) on card paper, imagining what they would look like when folded and taped together (see Appendix 1) In working on the models, some reasons for the uniqueness of the solids will emerge. This is a good time to start investigating why. This has mainly to do with the gaps between the faces of the solid, on the unfolded drawings. This method involves very intuitive ideas, and leads to a picture-based or geometric reason for the uniqueness of the five solids. The process of thinking goes along these lines:

**Plate II.2 Constructing dodecahedra**

1. Look at the model of a tetrahedron. In a regular tetrahedron, there are 3 equilateral triangles meeting at each vertex. What is the sum of the angles at each vertex?

2. Is it possible to have 4 equilateral triangles meet at a vertex of a regular polyhedron? If you put 4 equilateral triangles together at a point, could you fold the resulting pattern into a vertex? What is the sum of the angles at each vertex?

3. Repeat the above step for 5 and 6 equilateral triangles meeting at a vertex. Would it be possible to have more than 6 equilateral triangles at a vertex?

4. Can 3 squares meet at a vertex of a regular polyhedron? What is the sum of the angles at a vertex? Can 4 squares meet at a vertex? Explain your answer.

5. Follow the same reasoning for pentagons and hexagons. Try to extend to regular $n$-gons, where $n$ is greater than 6. That is, can 3 regular $n$-gons meet at the vertex of a regular polyhedron, if $n$ is greater than 6? (Remember that at least three faces have to meet at a vertex).

This method is, in essence, the one that Euclid uses in his "Elements", Book 13.

Although we have reasons for the uniqueness of the solids, we have yet to put them together to make a reasonable 'proof'. We have all the facts needed, but they are in a jumbled order, and have to be fitted together correctly, almost like a jigsaw puzzle. The idea of proof in mathematics is a very elusive one, and so far no one has come up with a really satisfying definition of what it is. In our experience, for the students, the idea of a proof was seen as being basically unnecessary - they saw it as simply a method of convincing someone else of a certain truth! (And perhaps it is no more than that!) To try to explain the idea, we used the following analogy: suppose that you wanted to make a dessert, say 'raspberry supreme'. Looking into your nearest cookbook, you see the following recipe -

"Add the sugar and stir well.
3 egg whites.
Crush the fruit with a wooden spoon until well mashed.
Chill and serve with whipped cream.
500 gms. ripe raspberries.
500 gms. sugar.
Beat thoroughly with an egg whisk till completely blended.
Whip the egg whites until stiff and fold into the mixture.
2 cups whipped cream."

A strange cookbook indeed! What would result if you were to carry out these instructions in the order given? For sure, some rather unpalatable conglomeration of eggs, sugar, raspberries and cream. Certainly not 'raspberry supreme'! In the same way, our facts were like ingredients, which had to be put down together with instructions in the right order to make a palatable (if not tasty) proof. And so we began with putting down our ingredients (facts about angles and geometry) and then gave instructions (inferences from combining these facts) in the correct order so as to reach a proof for the uniqueness of the five Platonic solids. This gave students a feel for what a proof consists of, and possible reasons for needing one.

The second method of proving the uniqueness of the five regular polyhedra is slightly more complicated, since it involves more algebra than geometry. In our project we went over this proof, although none of the students could later reproduce it. Perhaps if the students are a little more mature, it can be included in the project work. In any case, the exposure to such a proof is probably rewarding in itself.

The basic steps are as follows:

1. Let $n$ = number of edges per vertex, and $r$ = number of edges on each face. Note that $n$ and $r$ are both $\geq 3$.

2. Now every edge links two vertices, and so, $n\mathrm{V} = 2\mathrm{E} \Leftrightarrow \mathrm{V} = 2\mathrm{E}/n$

   Similarly, every edge borders upon two faces, and so, $r\mathrm{F} = 2\mathrm{E} \Leftrightarrow \mathrm{F} = 2\mathrm{E}/r$

3. We know, from Euler's theorem for solid figures, that F+V-E = 2

   By substitution, we have, $(2\mathrm{E}/r) + (2\mathrm{E}/n) - \mathrm{E} = 2$

$$\Rightarrow \quad (1/r) + (1/n) - 1/2 = 1/\mathrm{E}$$

$$\Rightarrow \quad (1/r) + (1/n) = 1/2 + 1/\mathrm{E}$$

   Or, the term on the left has to be greater than half.

4. Since $r$ and $n$ are at least 3, we look at possible combinations of $r$ and $n$ that satisfy the inequality $(1/r) + (1/n) > 1/2$. Note that both $r$ and $n$ cannot be equal to 4, since $1/4 + 1/4 = 1/2$; nor can they both take higher values.

5. This leaves just a few cases to check, since $r$ and $n$ are bounded above and below.

**Plate II.3 Displaying models**

**Possible extensions**

- The cube can be inscribed in the octahedron, and vice versa (join the centers of the faces of the cube, and the resulting lines form the edges of an octahedron). So also the dodecahedron can be inscribed in an icosahedron, and a smaller tetrahedron in a larger one (see Figure II.2.).

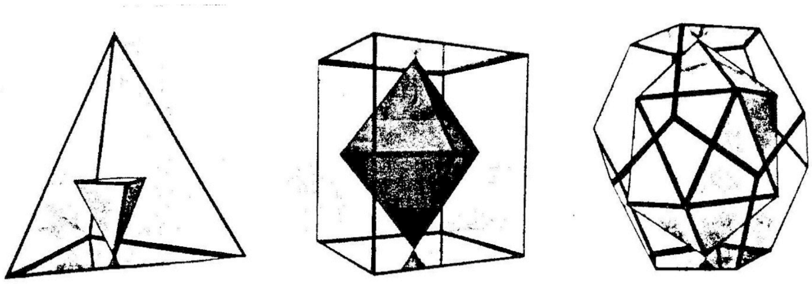

**Figure II.2.**

Reproduced from Harold R. Jacobs, *Mathematics: A Human Endeavor*, Page 217, [HJ]. © W.H. Freeman and Company. Reprinted by permission from W.H. Freeman and Company/Worth Publishers.

Depending on the age of the students, they can be encouraged to discover these inscribed solids without help and by their own investigations (perhaps by introducing graphs and their duality). If they are too young one can simply demonstrate these to the students with pictures and drawings, and enable them to visualize the inscribed solids. Many of our students were struck by these facts; they felt that this was part of some deeper pattern connecting the five solids.

- Painting on the platonic solids, for example, the world map on the icosahedron, tessellations of fish and other forms (by Escher) on the cube and dodecahedron. The painting should be done on the unfolded cardboard cut-out of the solid. Care must be taken to ensure that pictures on different faces will connect up when the cardboard is folded up to form the necessary regular solid (see Plate II.2).

- Soap bubble films using wire mesh models of the five solids. Dip the wire frames (suspended from a string) into a bucket of soap-solution and slowly remove them, being careful not to burst the bubble surfaces that are formed. The lines of contact between bubble faces make very interesting shapes and lead to much wild hypothesizing.

**Plate II.4. The Bamboo Icosahedron**

- Assemble a large model of an icosahedron, made of bamboo poles and simple metal joints (see Plate II.4.). When finished, the model can be roughly 4 meters in height and breadth. The icosahedron is an inherently stable structure, consisting entirely of triangles. Our students really enjoyed creating the structure, as it gave them a concreteness about the mathematical world that they did not, suppose existed! Besides that, it also gave them some experience with tools and construction.

- All five of the solids have been seen to occur in nature; certain microscopic sea animals called *radiolaria* are in the shapes of the dodecahedron and the icosahedron; crystals in the form of tetrahedra, and the cube (common salt), dodecahedron (Copper sulphate) and octahedron (chrome alum) are also found. Students can grow such crystals in the chemistry lab.

- There are many other activities and areas linked to this topic that can be studied, depending on the time and interest shown by the students. For example, there are the "semi-regular solids", i.e. solids with faces in the shape of more than one kind of regular polygon yet with every corner the same (see [HJ]).

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## III

# PROJECT -$\pi$[*]

*Now I, even I, would celebrate*
*In rhymes unapt, the great*
*Immortal Syracusan, rivaled nevermore,*
*Who in his wondrous lore,*
*Passed on before,*
*Left men his guidance*
*How to circles mensurate.* - A.C. Orr

($\pi \sim 3.14159265358979323846264338327 9$)

### Introduction

Mathematicians from the ancient to the most modern times have been fascinated by the constant $\pi$ (the ratio of the circumference of a circle to its diameter). Almost all the great mathematicians from Archimedes to Ramanujan have had a go at computing its digits. To this day the elusive hunt for its digits continues. Most high school students, when asked what $\pi$ is, either answer 22/7 or 3.14, and on rare occasions give its definition. Very few are aware that it is irrational, let alone transcendental[1]. Described below is a project involving an in-depth investigation into $\pi$ based on our experience.

### Objectives of the project

- Discuss the irrationality and transcendence of $\pi$
- Use various methods to compute $\pi$ using more and more sophisticated ideas, and give a flavour for the evolution of mathematical ideas
- Reinforce many of the topics studied using very concrete and specific research problems.
- Introduction to advanced topics not normally covered in the syllabus

[*] A brief version of this report on Project-$\pi$ was published in the proceedings of the Krishnamurti Centenary Educational Conference held at Rishi Valley School in October, 1995.

[1] A number is said to be *algebraic* if it satisfies a polynomial equation $a_0 + a_1X + a_2X^2 + \ldots a_n X^n = 0$, where all the coefficients $a_i$ are rational numbers. A number which is not algebraic is *transcendental*.

- Get a hands on experience of what it means to be involved in a classical problem that still concerns present day mathematicians
- To integrate the use of computers in the study of mathematics
- Learn to critique the methods used to evaluate $\pi$ in terms of efficiency, elegance and effectiveness.

**The Project**

**Irrationality and transcendence**

Start by understanding a proof that $\pi$ is a constant (we read through the chapter 'What is $\pi$ ?' from Lang's book [SL]). Lang starts by establishing that any circle has area $Ar^2$, where r is the radius of the given circle and A is the area of a unit circle. Denoting this constant by $\pi$, he goes on to show that the circumference of a circle of radius r will be $2\pi r$ and hence the ratio of the circumference of a circle to its diameter is a constant. Discuss the irrationality of $\pi$ (advanced students may want to see the proof), and what it means for a number to be transcendental. In this context you can talk about one of the great problems of antiquity - squaring the circle (see [SJ]). This naturally leads to a discussion on what it means for a problem in mathematics to be impossible to solve.

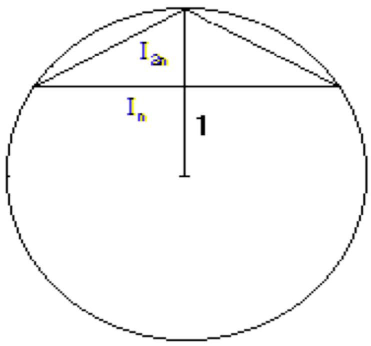
**Computing $\pi$**

In the next phase of the project proceed to study the techniques used by various mathematicians to compute $\pi$.

**Archimedes and Trigonometry**

The first mathematician to give a lower and upper bound for $\pi$ was Archimedes. In fact, many of the methods of computing $\pi$ are essentially the same as the one Archimedes used [WD]. Archimedes's method consisted of circumscribing and inscribing a circle with regular polygons. He first found an iterative formula for the length ($I_{2n}$) of one side of an inscribed polygon with 2n sides in terms of the length $I_n$ (see Figure III.1) of the polygon with n sides.

The formula he obtained was the following:




$$I_{2n} = \sqrt{2 - \sqrt{4 - I_n^2}}$$

**Figure III.1**

To derive this formula one needs only the Pythagoras Theorem. It is a good exercise for students to derive this on their own. An iterative formula for $C_{2n}$ (the length of one side of a circumscribed regular polygon of 2n sides) in terms of $C_n$ (the length of one side of a circumscribed regular polygon with n sides) is a little harder. In fact if we denote the length of one of the sides of an inscribed polygon by 'x', and that of the circumscribed polygon by 'y', then using similar triangles one can show that y and x are related by the following formula :

$$y = \frac{2x}{\sqrt{4 - x^2}}$$

Having established this relation we can get $C_{2n}$ in terms of $C_n$ from the earlier formula for $I_{2n}$. Once again it is a good exercise for students to derive this formula. So far we have used only high school geometry. Archimedes's main idea was that for a unit circle

$$n\, I_{2n} < \pi < n\, C_{2n}$$

Using a calculator and taking various values for n, starting with the well known fact that $I_6=1$ for a unit circle, we can get approximations for $\pi$. As 'n' increases one gets better and better approximations for $\pi$. Archimedes stopped at n = 48, and obtained the approximation $3.14084 < \pi < 3.142858$. It is quite mind-boggling to imagine that Archimedes made such precise calculations using the cumbersome Roman numerals. I understand that Bhaskaracharya using essentially the same method computed $\pi$ to be 3.1416 (using n = 384).

Using trigonometric formulae, for a unit circle, $I_n$ can be derived to be 2 sin $\theta$ and for the circumscribed regular polygon $C_n = 2 \tan \theta$, where $\theta$ is half the angle subtended by one side of the inscribed (circumscribed) regular polygon at the centre of the circle. Students can

see the power of a more sophisticated method to compute the same quantity much more easily. After understanding this method, and having the advantage of the calculator, students can be asked to compute the digits of π to about 10 decimal places.

We next examined the method that Newton used to compute π. Newton's method is probably one of the most ingenious methods for obtaining the digits of π.

We start with a semicircle (Figure III.2) with equation $y = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$,

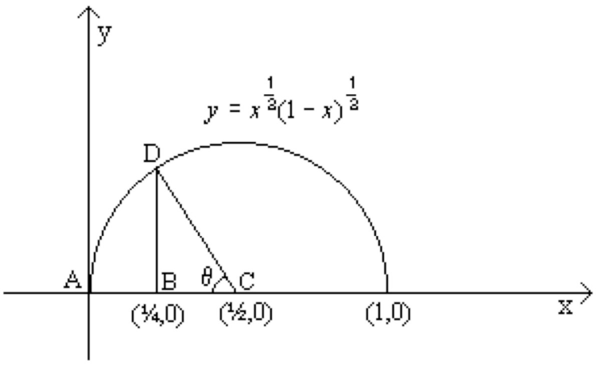



**Figure III.2**

and show using high school geometry that the area of ABD is $\frac{\pi}{24} - \frac{\sqrt{3}}{32}$.

Now using the binomial theorem (expanding up to the ninth term):

$$y = x^{\frac{1}{2}}\left(1-x\right)^{\frac{1}{2}} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} \ldots \frac{7}{256}x^{\frac{11}{2}}$$

using integration, the area of ABD is $\int_0^{\frac{1}{4}} y\,dx$; which is

$$\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} \ldots \frac{5}{704}x^{\frac{11}{2}} \ldots x^{\frac{13}{2}}\right]_0^{\frac{1}{4}} = \frac{1}{12} - \frac{1}{160} - \frac{1}{3584} \ldots \frac{429}{163208757248}.$$

Now equating the area obtained for ABD in two different ways we get π ~ 3.141592668. Newton expanded the above binomial expression only up to nine terms and obtained π correct to 7 decimal places. After gaining familiarity with the technique, students can be encouraged to use it to compute the digits of π, using calculators, this time to about 20 decimal places.

**Gregory, Madhava, Leibnitz, Euler and Ramanujan**

Students can then turn their attention to some of the familiar series, like the Leibnitz-Gregory-Madhava series ($\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$), that converge to π. They can also explore (without getting into proofs) other formulas and algorithms given in terms of series [PB], to generate π. Here, whenever possible, students can be encouraged to write computer programmes to compute π. Students begin to see that mathematicians are constantly in search of more and more efficient algorithms that yield the digits of π. Our students who did this project were in the 10th grade; and they found the series formulae, especially those of Ramanujan, a bit difficult to handle. Perhaps one could do this with a more advanced group.

**Buffon and probability**

Finally one can compute π using probability [B, pg. 161-165]. It is quite amazing that this constant should show up in an area that deals with chance. Buffon in 1777 posed the following problem. Suppose a needle of length 'L' is thrown at random (any position and any orientation are equally likely) on a ruled paper with parallel straight lines (the distance between the lines 'd' is assumed to be smaller than L), then what is the probability that the needle will intersect one of the lines.

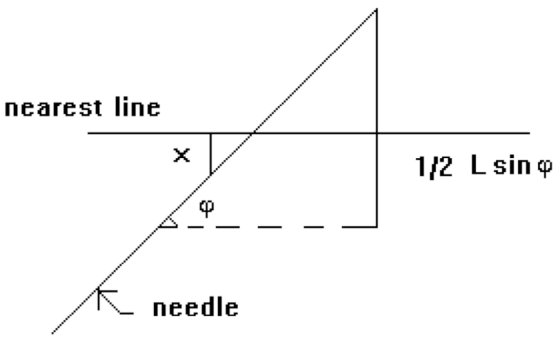



**Figure III.3**

Let 'x' (Figure III.3) denote the distance from the centre of the needle to the nearest line, and φ the orientation of the needle. Since we measure x from the nearest line it suffices to consider just one line. From the above figure it is clear that the needle will intersect the line if and only if $x < \frac{1}{2} L \sin \varphi$. The problem then is to find the probability P ($x < \frac{1}{2} L \sin \varphi$).

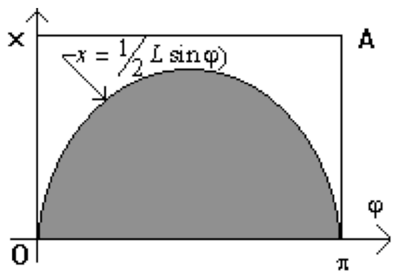



**Figure III.4**

Consider a rectangle OA with rectangular coordinates x, φ (Figure III.4). The points inside this rectangle satisfy the inequalities $0 < x < d/2$ and $0 < \varphi < \pi$. Then each point in the rectangle corresponds to one and only one possible combination of position (x) and orientation (φ) of the needle. Since all such combinations are equally possible, the area of the rectangle represents the sum total of all possibilities that can arise. Further the needle will intersect a line when $x < \frac{1}{2} L \sin\varphi$, that is positions and orientations corresponding to points under the curve drawn above. The area of the curve above is given by $\frac{1}{2} L \int_0^{\pi} \sin\varphi \, d\varphi$. Hence $P = \frac{2L}{\pi d}$. Therefore $\pi = 2L / Pd$, and taking $L = d$, we get $\pi = 2/P$. This is a completely new way of computing π. 'P' can be computed by throwing a needle on a ruled paper many, many times and recording the ratio of the number of times the needle cuts a line to the total number of throws.

Here again students can be encouraged to write an algorithm to simulate the experiment of needles being thrown on a ruled paper to compute the digits of π. Although the method in itself was interesting, students discovered that this method was really poor when it came to computing the digits of π ( see [PB] for a table listing number of throws versus the value of π). To understand why, imagine tossing a coin 10, 100, 1000, or 10,000 times. The actual number of heads that turn up will differ from the expected 5, 50, 500 and 5,000. This difference is called the absolute chance error and you can see that the error is bound to increase with the number of tosses. In the needle experiment, the accuracy of π depends on the probability of obtaining 'k' correct decimal places in 'n' throws. This probability is very small just as it is very unlikely that you will get exactly 5,000 heads in 10,000 tosses of a coin.

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# IV

# EXPLORING FRACTALS

*" L`imagination se lassera plutot de concevoir que la nature de fournir."*
(Imagination tires before Nature) - Blaise Pascal

**Introduction**

The geometry that is taught in high schools is normally Euclidean Geometry. While Euclidean geometry is beautiful in its own way and is perhaps the first place where students experience how modern mathematics is done, for many it is a dry subject, which fails to describe a cloud or a tree or a sand bed. As Grunbaum and Shephard put it,

*" At the high-school level it has long been the tradition to use geometry as a vehicle for teaching logical reasoning and the deductive method, without much regard for the geometrical content. ... the essence of the subject - its visual appeal - has been completely submerged in technicalities and abstractions."*

In fact Euclidean geometry fails to provide models which simulate many of the patterns that occur in nature.

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In the late 70`s Benoit B. Mendelbrot invented a new geometry that would respond to these questions and called it Fractal Geometry. The technical definition of a fractal is a bit complex and will be given later. But in Mandelbrot`s own words "A fractal is a shape made of parts similar to the whole in some way ". The word *fractal* is derived from the Latin adjective *fractus*, which means irregular and fragmented. I must say here that in fact a fractal although irregular is far from fragmented in the normal sense, but is in many ways holistic. As we shall see, inherent to the concept of a fractal is the notion of self-similarity.

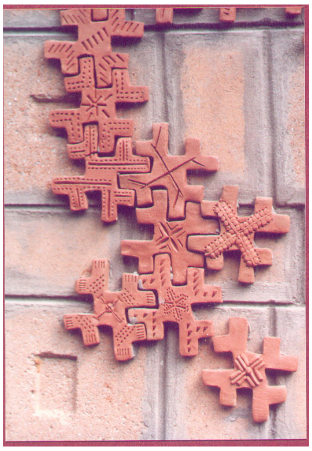


**Plate IV.1 A fractal that tiles**

Fractals come in all kinds of shapes and forms, some are monster curves, some are natural forms and some are mere dust! The topic of fractals, not normally treated in high school curricula, is perfect for a project in mathematics. It can be introduced to students who have had mathematics only up to the 10th grade and it is a chance for students to see mathematics created in this century and also experience directly the visual beauty of nature and mathematics. The time spent on the project can be two to three hours a week for a term (3 months).

**An introduction to fractals**

First expose the students to a variety of fractals, followed by an introduction to the notion of a fractal dimension and then give them the formal definition of a fractal.

The best fractal as always to introduce to students is the famous **Koch Snowflake curve** (Figure IV.2) introduced by the Swedish mathematician Helge von Koch in 1904. The curve is obtained by starting with an equilateral triangle and replacing each side with the following *replacement rule* (Figure IV.1)

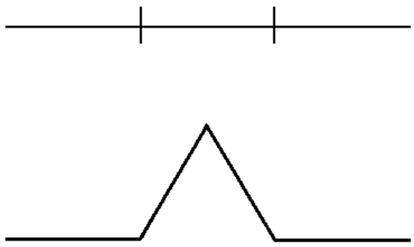


**Figure IV.1**

and then continue this process infinitely many times to obtain the snowflake curve.

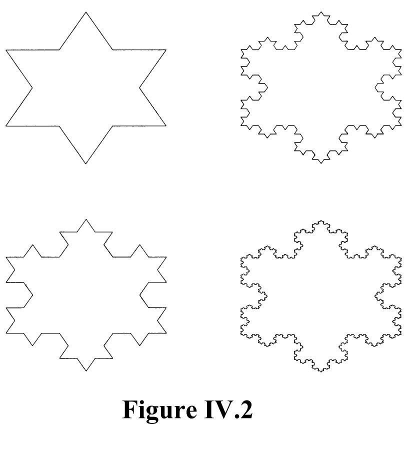


**Figure IV.2**

It is a nice exercise for students to calculate the perimeter and area of each new iteration in terms of the previous figure. It is also a mathematical curiosity that while the perimeter of the curve tends to infinity the area it bounds is finite!

There is a three-dimensional analogue to the snow flake curve. You start with a tetrahedron and on each of its faces you attach a smaller tetrahedron whose edges have half the length of the original one and repeat this process to get each new iteration.
The question is, what kind of figure do we expect to emerge after many iterations? Most mathematicians would guess that it is an infinitely bumpy surface. But the amazing thing is that it will tend to a cube and this is apparent within four iterations (see Plate IV.2).

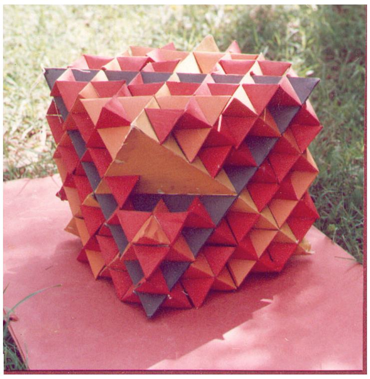


**Plate IV.2 Three dimensional Snow Flake Curve.**

Apparently the human lung is very similar to this fractal, the exterior of the lung is smooth, but inside there is a mass of tubes that branch into smaller and smaller tubes. There are many variations to this theme and one can start with an octahedron instead of a tetrahedron (see [DC] for details).

If one starts with a line instead of a triangle we get the **Cantor dust set.** Start with a line, divide it into three parts and get rid of the middle third. We then do the same to the two remaining segments and so on. Here we have obtained a set of points, rather than a curve as a fractal. Anyone who has done advanced Calculus will recognizes the above fractal as the Cantor set, which provides a host of counter examples in mathematics.

All of the above are examples of what are referred to as edge substitution fractals. Another famous example is the **Quadric Koch Curve** (Figure IV.3, Plate IV.1, Plate IV.4). Here you start with a square and replace each edge with the following replacement rule to produce this beautiful fractal.

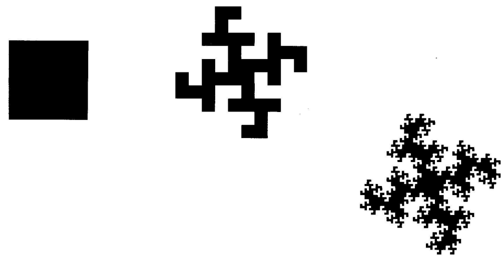


**Figure IV.3**

Reproduced from Benoit B. Mandelbrot, *The Fractal Geometry of Nature,* Plate 53, Page 53, [BB].© Benoit B. Mandelbrot. Reprinted by permission from Prof. Mandelbrot.

Obviously this rule is not unique and students were encouraged to experiment with different possible replacement rules. This fractal proved to be a hot favourite among students as will be discussed in the section on projects. The astute reader would have also recognized that to create an edge substitution fractal one has really used the 'nibble' technique described in the article on tessellation, and hence they tile a plane.

The reader would have also noticed that fractals seem to have the property that at each new iteration they are similar in appearance and scale to the previous iteration. This property is referred to as self-similarity and is a key characteristic of fractals.

The next fractal that was studied was the **Seirpinski gasket** (Figure IV.4). Here one starts with a triangle and one joins the midpoint of each side to produce four triangles inside the original one.

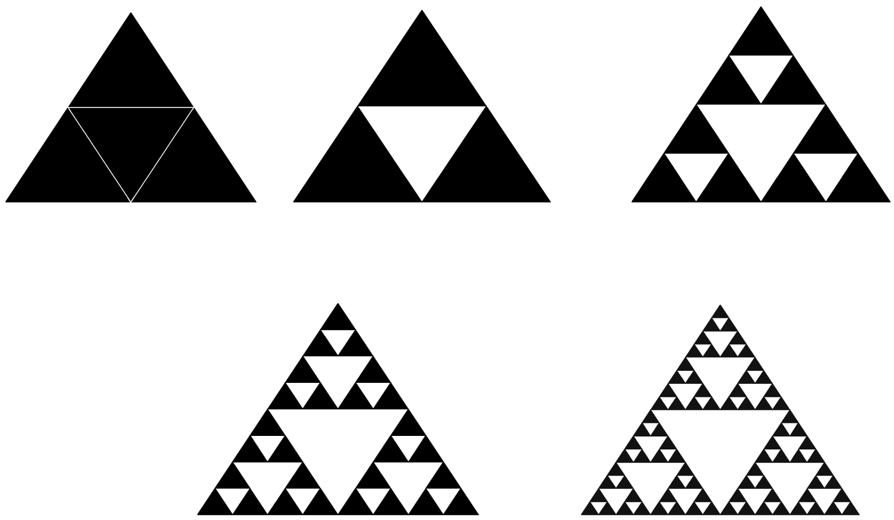


**Figure IV.4**

Then somewhat similar to the Cantor set one colours the middle triangle black (or removes it). Proceed in the same manner with the remaining three triangles and repeat this process ad infinitum to obtain a fractal. One can produce a mountain-like figure by adding some randomness to this process.

The Seirpinski gasket can also be produced in another amazing way. Fix three non-collinear points in a plane, say A,B,C. Pick any point at random in triangle ABC and call it $P_0$. Choose any one of the three vertices A,B,C at random, say A. Find the mid point of $AP_0$ and name it $P_1$. Keep generating points in this fashion to produce the Seirpinski gasket! It is not hard to get a computer to generate this fractal and students wrote programmes to do so. Obviously the starting figure for the gasket can be varied to include figures like a square etc.
There are also three dimensional analogues of the gasket and they really look like extra terrestrial objects (Figure IV.5).

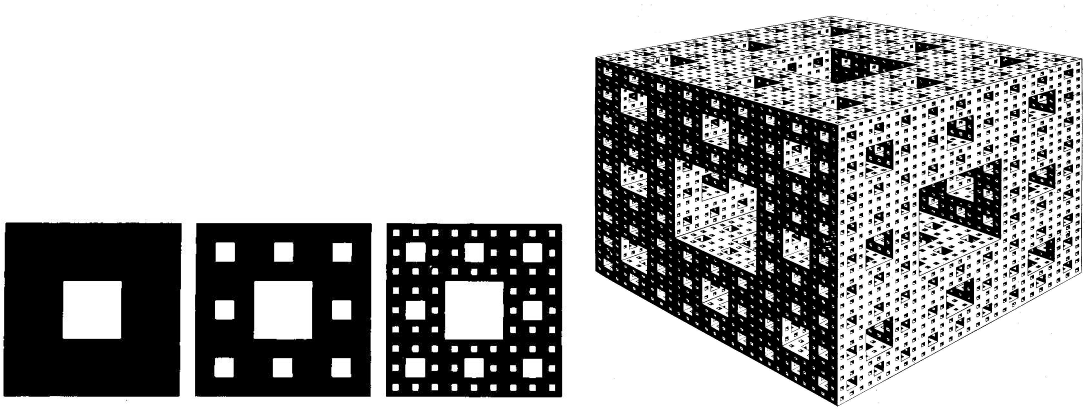


**Figure IV.5**

Reproduced from Benoit B. Mandelbrot, *The Fractal Geometry of Nature,* Plate 145, Page 144, 145, [BB].© Benoit B. Mandelbrot. Reprinted by permission from Prof. Mandelbrot.

As a digression from regular fractals one can discuss the notion of **Space Filling Curves** i.e., a curve that completely fills two dimensional space. Consider the famous Peano's space filling curve. Here again one starts with a square and replaces each edge with the following replacement rule.

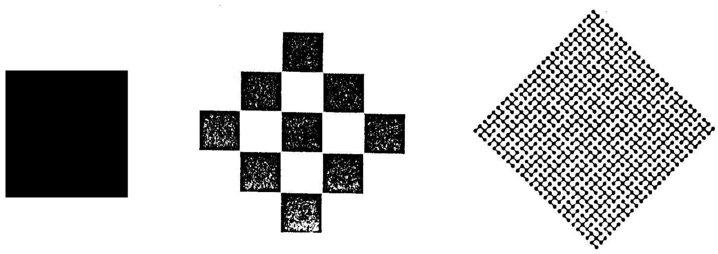


**Figure IV.6**

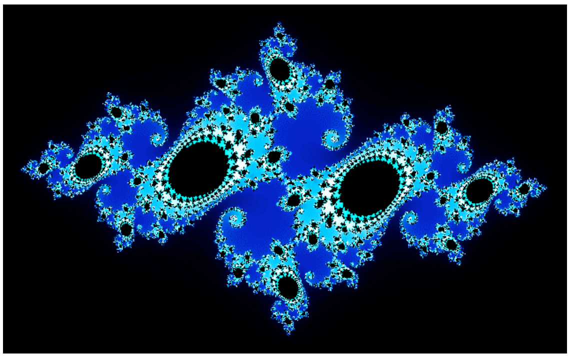
Reproduced from Benoit B. Mandelbrot, *The Fractal Geometry of Nature,* Plate 63, Page 63, [BB].© Benoit B. Mandelbrot. Reprinted by permission from Prof. Mandelbrot.

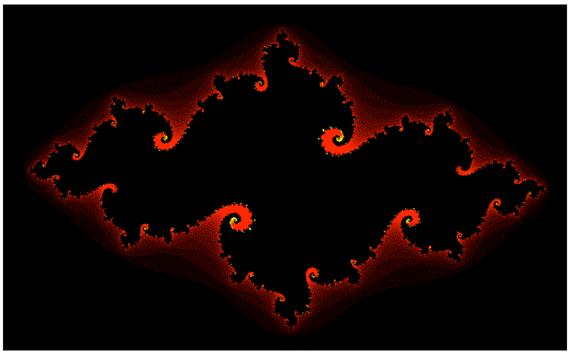
After several iterations this ‘curve’ completely fills the square. Once again it is not hard to write a computer programme that can generate this figure and get students to experiment with different possible space filling curves.

The most startling and beautiful fractals students get to learn about are the **Julia Sets** and the **Mandelbrot Set.** Here one needs to introduce students to a little bit of Complex numbers. The basic idea is as follows. Consider the Complex iterative formula

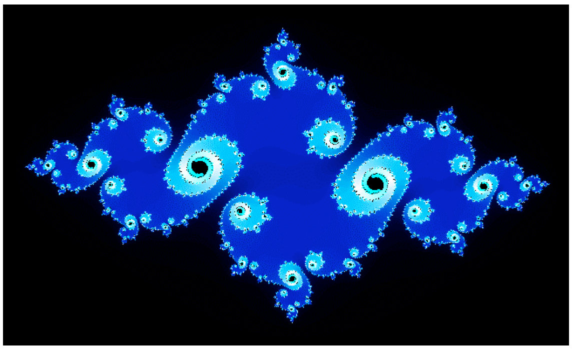
$$Z\ (\text{new}) = Z\ (\text{old})^2 + C$$

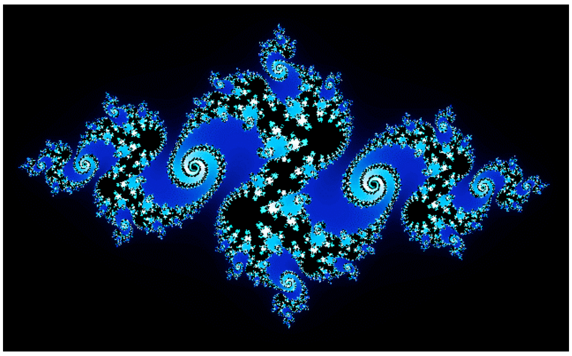
for Julia sets we consider $|Z| < 2$. The value that Z tends to is called the attractor point for Z. If the point is attracted to a point with $|Z| < 2$ we say it is attracted to zero and if it goes to a point with $|Z| > 2$ then we say it is attracted to infinity. We take different starting values for ‘C’. ‘C’ is the key. Different C’s produce different Julia Sets. Now if Z is attracted to infinity the point gets coloured white and if Z is not attracted to infinity it gets coloured black. It is really incredible that such a simple procedure produces such visual treats.





**C = -0.748 + 0.096i** **C = -0.736 + 0.096i**





**C = -0.758 + 0.096i** **C = -0.76635 + 0.09664i**

**Figure IV.7 – Julia Sets**

These images were created using WinFract 18.21

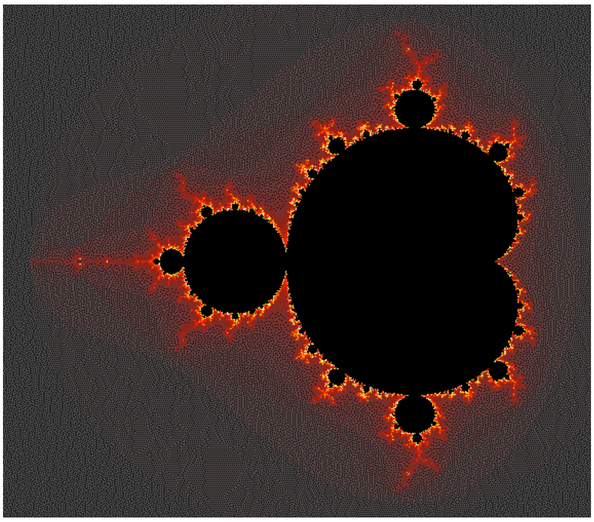


**Figure IV.8 – Mandelbrot Set**

This image was created using WinFract 18.21

For the Mandelbrot set, once again one starts with

$$Z\ (\text{new}) = Z\ (\text{old})^2 + C$$

but always starting with Z = 0. This is one of the most fascinating and beautiful mathematical objects to have been created. Most computers nowadays come with software to generate this fractal, and as one zooms closer and closer into the set, one gets figures which are self-similar and truly amazing pictures.

**Fractal Dimension**

After giving students a feel for the various fractals, one can introduce them to the idea of the fractal dimension. Till now we have not defined a fractal rigorously. To do so we need the idea of the fractal dimension. This notion was the outcome of a fairly innocent question: "How long is the coast line of Britain?" It seems at first a fairly simple measurement to carry out, but readers may be amazed that the common frontier between Spain and Portugal, as reported in these neighbour's encyclopedias, differ by as much as 20%. This discrepancy arises because when measuring a border such as between two countries, or the coast of Britain, we make crude estimates about the squiggles. If we were to be more fussy, then our measurements would yield different results depending on how careful one is. Another unexplained paradox is that for the snowflake curve, its perimeter tends to infinity, while it encompasses only finite area. So clearly there is a need for a different way of measuring the borders of fractals. Mandelbrot introduced the fractal dimension or the Housdorff-Bosovitch dimension, defined as

$$D = \frac{\log(N)}{\log(1/r)}$$

where ‘N’ and ‘r’ come from the replacement rule for the fractal or also referred to as the generator. So for example in the snowflake curve we started with a segment of unit length, then we divided it into three (b in general) equal parts using a scaling factor of 1/3 (r = 1/b) and used 4 (N) such pieces to create the generator. So for the snowflake curve D = log 4/log3 = 1.26...

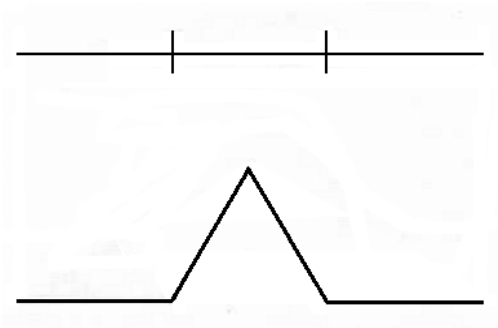


**Figure IV.9**

A regular figure such as a square, is scaled by a factor of 1/n and we need $n^2$ such pieces to produce the full square, so here D = 2 log n / log n = 2. So in this case the fractal dimension and the Euclidean dimension coincide.

In the case of the Seirpinski triangle it is an easy exercise to see D = log 3 / log 2 = 1.58.... Readers might wonder how any dimension can be fractional. This is part of the wonder of fractals. One way of interpreting this dimension is to think of it as a measure of how much space the border occupies. So in the case of the snowflake curve having fractal dimension greater than one makes sense.

For a long time an open problem was: “ What is the fractal dimension of the Mandelbrot Set ?”. In 1991 Shishikura a Japanese mathematician showed that it is 2. To quote Milnor, a famous topologist, “This means the boundary is as thick as it could get without occupying an area”.

We are now finally ready to define a fractal! *A fractal is a set whose Hosdorff-Besicovitch dimension is strictly greater than its topological dimension.*

## Projects and Activities in the Classroom

In most books and manuals on fractals, the main area where activities and projects are explored is the writing of computer programmes to generate fractals. This however leaves a large majority of students uninterested. In fact fractals is a topic which can appeal to many students. Hence we found various activities other than computer projects to help students discover fractals around them. We list the various projects students can get involved in.

**Plate IV.3 Fractal sand bed photographed by students of CFL**

Fractals in Nature: Students can go in search of fractals in Nature in and around the school. They can be encouraged to find fractal like trees, sand beds and ferns. It was very exciting to recognize the Rain tree, Subabul and Fir tree as fractals. Ferns exhibit amazing levels of self similarity. A common household fractal is the cauliflower which also exhibits self-similarity to a very high degree and it is quite amazing to see how the whole shape is contained in each floret. Students can take photographs of their fractals and display (Plate IV.3) them in imaginative ways. This project helps students get a concrete feel for the ideas of self - similarity and the notion of the replacement rule.

**Plate IV.4 Poster of Quadric Curve.**

Posters and three dimensional models: Students with artistic inclinations can really enjoy creating posters of fractals. Favourites were collages of the Seirpinski gasket and the quadric curve (see Plate IV.4). A few students made a model of the three dimensional Van Koch curve (see Plate IV.2). It was truly amazing to see the frame of the cube appear within four iterations and this remains one of the masterpieces of the mela. The model required great care and patience. Other students can explore making fractals in clay. Our students made a terracotta mural of the quadric curve. They first made a metal stencil of the curve and cut out clay slabs using the stencil. The clay tablets were baked and as we know, these tile and a beautiful mural was mounted on one of the classroom walls (see Plate IV.1).

Fractals and the Computer : Students with an interest in computers can write programmes (our students used mainly Quick Basic) to generate edge substitution fractals, tree fractals and the Mandelbrot set.

Fractals in the Lab : Students can also create fractals in the Physics and Chemistry labs. When you electrolyze lead bromide the lead begins to grow on the cathode and the growth is tree like in nature because of diffusion from the surrounding. The same is true for crystal growth in a crystal garden. This involves placing crystals like Copper Sulphate, Ferrous Sulphate, Nickel Sulphate in a solution of Sodium Silicate (water glass). The salt dissolves and diffuses outwards and the insoluble silicate precipitates out, forming tree like growths.

Advanced topics: Explore the use of transformations to generate fractals (see [TB]), give a rigorous proof that the three-dimensional version of the snowflake curve discussed above will tend to a cube, experiment with other three dimensional snowflake curves [DC] and begin to get into Chaos theory.

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# V

# CHANCE AND STATISTICS

*"If all the statisticians in the world were laid end to end...it would be a good thing!"*
- Anonymous

Statistics is usually assumed to be a dry and boring subject, and indeed whenever you read a section on statistics in a high school text, it confirms your worst fears. Calculating the mean, median, mode, standard deviation, ranges, etc. are mechanical jobs at best, and unless a student is actually curious about the nature of the data and the variables, he/she can't be expected to get excited about such things. So the first objective for the teacher is to find a data set, or better yet get the students to collect their own data, so that they will be eager to summarise it in many ways, and to get as much meaningful information as they can out of it.

The other topic taught in connection with statistics at the high school level is probability, and here the big enemy is fear! Most students (and some teachers too) go completely blank when they see a problem like:
"A committee of 8 members consists of one married couple, 4 men and 2 women. A working party of 4 is to be chosen. What is the probability that the party will not contain both husband and wife? If the 8 members are to sit around an octagonal table, what is the probability that the man is sitting next to his wife? That the 3 women are sitting together?"
Yet you can think of probability as common sense, and when students approach such problems confidently, applying the fixed set of rules that they know well, they often surprise themselves with their problem solving capacities.

During our Math mela, we attempted to address both concerns in a creative way. Here is a description of what you could do with your students based on our work.

**Statistics**

To make statistical calculations more interesting, ask students to collect data on the reaction time to a simple task - catching a strip of cardboard between thumb and forefinger when someone releases it just above your hand. Children of different ages can be involved at different levels in this investigation. 10-12 year olds can collect the data from people they know – perhaps ten people each. Each child makes a 40 cm long strip of cardboard with

centimetre markings down the length. The reaction time is measured by the cm marking at which the person catches the strip. This is an example of 'operationalising' the variable reaction time – and students must understand that it is just one possible way of doing this. You can introduce the idea of variables and the importance of standardising data collection methods. Let them do the experiment on each other a few times, and note the variables that affect the results. Some of these are – the distance between the bottom of the strip and the person's hand, whether you give warning to the person or not before dropping the strip, etc. They must realise the importance of keeping these things constant from one trial to the next, for each person tested. The basic experiment can be repeated eight times with each person, rejecting the first three trials as unreliable measures. You can explain why the average of the remaining five values is a better (more reliable) measure of the person's 'true' reaction time. In addition, students can note down a few other simple variables such as sex and age of each person. The data must be tabulated neatly. The learning objectives for this task would be:

1. to understand how standardising testing procedures removes bias from the data.
2. to observe variation in data - both within a single person as well as among different people.
3. to use the mean of repeated trials as a way of getting a more reliable estimate.

Slightly older children, 12-14 year olds, can take this experiment further by learning how to plot the data in a histogram, summarising it by calculating the average, median and mode. You can introduce measures of spread (the standard deviation and range) as well. The histogram should roughly approximate a bell-shaped curve, and the ubiquity of this shape can be discussed. The data can then be split into two subsets – males and females – histograms drawn, means and S.D.s calculated, and comparisons made between the two halves of the dataset. They can draw a scatter plot of the relationship between age and reaction time, which may or may not yield a pattern. Learning objectives for this age group will be more or less those of any high school chapter in basic statistics.

Older students with a particular interest in computers can learn the use of a statistical computing software package (we used S+ for PC). They can enter the data, write simple programmes to calculate the summary statistics, draw histograms, scatterplots, and then make a few comparisons. For example, they can calculate the correlation between reaction time and age, and compare the average reaction times of males and females.

**Probability**

Everyone is attracted by the idea of gambling - even those who have never been inside a casino have some romantic ideas about it - and high school students are no exception. The idea of gambling and risk taking seems only to excite students, and although gambling is as full of probability as their text books, few students realise it. So you can introduce them to fairly complex concepts in probability through this topic. This project is suitable for 11th and 12th standard students.

We chose to play the classic roulette, where a circular disk is divided into 38 numbered equal segments – 18 red, 18 black and two green (numbered 0 and 00). The disk is spun, and comes to rest with an external pointer pointing to a segment at random. There are many bets one can 'place' – for instance, you can bet that the segment will be red, or an even number, or a particular number like 12. No bets can be placed on the green segments (in fact this is what gives the 'house' its advantage!). For details of the game, look up an encyclopedia!

The students must calculate the probabilities of winning/losing on each bet, and fix odds for each that would benefit the house while still holding out enough hope for players to be attracted. This is quite a subtle job, but easy once you have taught them the basic concept of expected value. They can then calculate how much the 'house' could be expected to win in the long run, give or take how much. Although this task was quite advanced, our 16 year olds were able to understand terms like 'expected value', 'standard error', 'long run average' etc. in a real setting.

For the dice game, students can calculate the probabilities of obtaining at least one ace in four rolls of a die, and that of obtaining at least one double-ace in 24 rolls of a pair of dice. This is a problem from seventeenth century French gambling (and mathematics!). In those days, it was thought that the two events were equally likely, but experience showed the gamblers that the former event occurred slightly more frequently than the latter. They wrote to the famous mathematician Pascal with their question, and he solved the problem with Fermat's help. He calculated that the first probability is 51.8% while the second is 49.1%! Have your students make this calculation, but only after playing with dice for a while, taking empirical evidence to begin with and getting a feel for the probabilities. For the calculation, you will need to introduce fundamental topics like addition and multiplication rules for probability, mutually exclusive and independent events.

# VI

# RECREATIONAL MATHEMATICS

*A good mathematical joke is better, and better mathematics, than a dozen mediocre papers*

- J.E. Littlewood

**Plate VI.1 Collapsing Sphere**

## Introduction

The quote from Littlewood more than justifies the need to have recreational mathematics as part of a school mathematics curriculum. In this chapter we discuss how puzzles, games and treasure hunts can be part of doing mathematics in school.

## Puzzles, puzzles and more puzzles

Since most puzzles have no formal requirement of a mathematical background, every one feels confident about having a go at solving them. Children have an opportunity to try their problem solving skills without thinking of it as a test of their understanding of a given concept. This allows them to play with ideas and solutions – a prerequisite to enjoying mathematics! The three dimensional puzzles really challenge the students' spatial sense. It is interesting to discover spatial competence in some children who have not previously taken to academic mathematics.

We now offer suggestions on how puzzles can become part of school life.

- Create a softboard devoted to mathematical recreation. Students and teachers can pin cartoons, puzzles and any other material relevant to mathematics. Put up solutions at regular intervals, and teachers can discuss solutions in their classes.

- Have a whole school assembly devoted to mathematics. It is quite an interesting experience doing mathematics across the board (from age 6 -60!) with the whole school. Often younger

children will attempt difficult problems without a sense of inhibition. The assemblies can also provide an opportunity to present famous problems like the four colour problem[1] and the Königsberg bridge problem[2].

- Get students to make mathematical puzzles, both two- and three-dimensional ones. It not only forces them to make use of many mathematical concepts that they have already learnt, but also to learn what it is to make something that is aesthetic and useful. We have included, the collapsing sphere (see Plate VI.1, and Appendix 2B), and a topological puzzle (see Appendix 2C). Other ideas are to create the tower of Hanoi, which consists of three pegs on a flat base, and the middle peg has seven discs whose sizes increase from top to bottom. The idea is to shift all seven discs, one at a time, from the middle peg to one of the outer pegs, always making sure a smaller disc is above a larger one. You can also make the soma cube (see Plate VI.2), consisting of

**Plate VI.2 Soma Cube**

seven wooden pieces that can be assembled to form a cube. It has many solutions.

- Introduce children to 'folk mathematics'. Folk mathematics is a collection of all the traditional songs, riddles, games and designs that are used in daily life. There may be no formal structure or definition or no derivations taught, but there is an empirical use of numbers and patterns to help either in daily activities or purely for entertainment. For example- a hundred people were invited for a feast and a hundred *papads* were fried. Each man was served three *papads*, each woman two and each child half. How many men, women and children were present at the feast? Younger children solved these riddles empirically but the older ones used algebra to solve them. We have included puzzles taken from villages around Tamil Nadu from [RRS] (see Appendix 2D).

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[1] the four-colour problem requires one to prove that every map on a flat surface or a sphere can be coloured without using more than four colours. The only requirement is that no two countries sharing a common border should be the same colour.
[2] the problem is to decide whether it is possible to take a walk in the town of Königsberg in such a way that every bridge in the town is crossed once and only once and the walker returns to the starting point.

- Let the students explore the art of Kolam or Rangoli (an artistic design found on the floor in a place of worship or at the entrance to a house). Kolams are usually drawn with dots – either around them or by joining them, and are generally symmetrical. Some are single line kolams that can be drawn without lifting pen from paper. Designs can be enlarged by proportionately increasing the number of dots (see Figure VI.1). With a grid of five rows of five dots each, say, a hundred thousand different designs can be drawn!

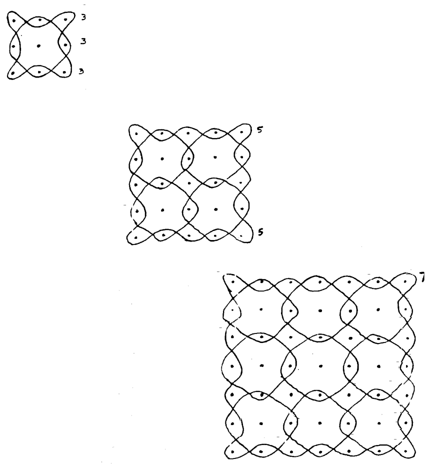


**Figure VI.1 Kolam drawings**

Reproduced from A. Rampal, R. Ramanujam and L.S. Saraswati, *Numeracy counts!* [LS]. National Literacy Resource Centre, Mussoorie, 1998. Reprinted by permission from L.S. Saraswati.

- If sufficient enthusiasm has been generated and many children have created puzzles and games, then as we did, have a *'Santhe'* (see Chapter VIII for details).

**Mathematical Games**

Board games have fascinated mathematicians for a long time and now we even have an area of mathematics called game theory. So it seems redundant to justify the need to encourage students to play and create mathematical games. Games can help younger children memorize concepts and to master skills in an enjoyable manner. It also encourages group interactions and cooperative learning. For example, we used a variation of the traditional tic-tac-toe game (see Appendix 2A). Apart from learning the usual strategies of how to win a tic-tac-toe game, younger students get to practice their addition. As the children play game after game, they are able to perform simple additions with amazing speed. It also seems important that children be involved in all aspects of making the game – from constructing the board and making it as colourful as possible (see Plate VI.3) to learning the rules. This allows children of varied mathematical abilities to have equal involvement.

**Plate VI.3 Board Game made by students of CFL.**

During the Math Mela we created a game called the Stock Exchanger. In the next section we discuss the game and its objectives in detail.

## THE STOCK EXCHANGER: A stock market game

**Introduction**

Stock Exchanger is an interactive educational game. The teaching objective is to inform the students about various aspects of the real-life business world in general and the workings of the stock market in particular. Along the way students also get to use mathematical concepts of percentage, profit and loss, graphing and also how to calculate dividends. These concepts are usually part of the commercial mathematics portion of a 10th Std. syllabus. The objective of the game for the student is very straightforward: to maximize earnings at the end of one week's trading, starting from an initial capital of Rs. 10,000/- given to all players.

**Pre-game requirements**

The teacher holds a two-hour workshop before commencement, explaining the workings of the stock market. The roles of expectation, risk management, as well as the returns for investment outlined in detail. Concepts such as dividends, debentures, real interest, etc. are elucidated.

**Procedure/rules:** The teacher/ game master selects or creates ten companies trading on the school stock exchange. The list of companies in the CFL Stock Exchange along with the par values of the shares were:

1. Allied Shipping - 10
2. Modi Pagers - 10
3. Sony-10
4. Titan-10
5. Coorg Coffee House - 10
6. Quatar Oil - 100
7. Enron - 100
8. BFL Software - 100
9. Doordarshan - 100
10. HongKong Bank - 100

# The CFL Times

**Volume 1, Issue 3** **July 1995**

### Initial response to Sony CTVs puts pressure on production

New Delhi 5th July

A year after its entry into India, Sony India Pvt Ltd., the fully owned subsidiary of Japan's Sony Corporation, is ready to launch its primary product – the 21-inch colour TV.

After the launch of the 21-inch CTV, Sony expects to introduce a 14-inch model and a 25-inch model, giving it a complete line of CTVs. The company's game plan is to sell 300,000 CTVs in India by 1998, and one million by the year 2000.

### CM Gheraoed, Police Cane Demonstrators at Madikeri

DH News Service

Police caned members of the Kodagu district Joint Agitation Committee of coffee growers who gheraoed Chief Minister Deve Gowda for 30 minutes in the court compound at Madikeri today. Hundreds of demonstrators showed black flags in protest against the latest tax imposition from March 23rd. The CM said later that he was angered by the unfortunate incident and vowed to stick with the government policy.

### Titan to Boost Watch Output at Two Plants

Reuters, Bombay 5th July

Titan Pvt. Ltd. will boost output this month at two plants to meet strong demand for its luxury watches, a Titan spokesman said.

Titan will buy four holidays from workers at its plants in Pune, and at Mohali, North Chandigarh.

On July 3rd, Titan had a 20,173 unit order backlog for its premier luxury product. When it launched the Tanishq series the monthly sales target was only 7000 units.

### Motorola to tie up with Reliance

Bombay 5th July

In order to capitalize on the liberalization of the telecom sector in India, Motorola incorporated has set up a joint venture with Reliance India to produce pagers and cellular phones.

Motorola is the world leader in cellular phones and pagers. Motorola also heads the Telecom sector in investment in R&D, which in 1994 alone was $1.86 billion.

The unit is said to go operational in four months.

### Allied Shipping Trade Union Strife may Intensify

Calcutta, 5th July

Even as the lockout called by the Allied Shipping management at the company's Calcutta unit enters its first day; signs are that Dr. Dutta Savant seems to have consolidated his hold on the unions.

Demonstrations are likely at the company's headquarters to start pressurizing the management.

He/she preselects the fortunes of these companies at the beginning, creating valid reasons for their rise or fall. These reasons are NOT made plain and must be surmised from articles in a daily ‘newspaper’ created by the game master and displayed on a notice board every day. See for example CFL Times volume (3) overleaf. The article entitled ‘Allied Shipping trade union strife may intensify’ would imply a downward trend in the present stock of ‘Allied Shipping’ and the article entitled ‘Initial response to Sony CTVs puts pressure on production’ would imply an appreciation in the ‘Sony’ stock. In fact, in CFL times Vol. 4, published the next day, Allied shipping stock was down 10% while Sony appreciated 14%. One may try using real stocks and the daily newspaper instead of fictional ones. The only disadvantage may be that dramatic changes in prices may not occur!

The player must first select a portfolio of stocks from the initial capital of Rs. 10,000/- he/she must then correctly decide the direction of his/her stocks and adjust investments accordingly. These changes are noted by the game master in a logbook everyday. All accounts are cleared in a half-hour period at the end of the day. By the end of the week (five rounds), the player with the best investments and the greatest gain wins. Players may also ask the reasons for the trends after the particular trading period has finished (i.e.-the next day).

**Mathematical Gains!**

- Concrete feel for shares, debentures, dividends etc.
- Learn to calculate percentage loss/gain for each company in their portfolio.
- Plot graphs reflecting the performance of the companies in their portfolio.
- Students start reading the newspaper to get a better feel for the business world.

## A MATHEMATICAL TREASURE HUNT

*... But the Snark is at hand, let me tell you again!*
*'Tis your glorious duty to seek it!* - Lewis Carroll

Every one loves a good treasure hunt; the joy of solving a clue, the thrill of locating the place where the next clue is hidden and the general race against time. Here we describe how a treasure hunt with mathematical clues can be created.

In traditional treasure hunts one clue leads you to the location of the next and so on, till you are finally led to the location of the treasure. How does one describe a location using mathematical clues? Well! I guess René Descartes had figured this one out! Create a map of the school on a grid. Each clue leads to a pair of numbers which when plotted directs the children to a place in the School's campus where the next clue is located. One difference in this treasure hunt is that the children have to come back and gather in an allotted classroom, where they can use a blackboard or paper to solve their clues. Also each clue should contain a letter in a different font. This way when children solve all clues they have a set of letters, which when unscrambled lead them to the treasure. Teachers can experiment with other co-ordinate systems like polar co-ordinates or even come up with ways of translating mathematics to language.

From our experience we can say confidently that children enjoyed the treasure hunt immensely. The structure forced them to think on their feet. Since the questions were quite out of context, they had to use all the mathematics they knew, without the option of referring to a textbook. Efficiency of solution was a key to success. Each location where children gather to solve should have a teacher who can give some hints if the students are hopelessly lost. This enables the treasure hunt to get over in a reasonable amount of time and also prevents undue frustration. Since each group had an age range of about 2-3 years, co-operative learning was fostered and students had to share their knowledge base.

The clues used for the middle school and senior school are given below.

## Instructions to the students

1. Each clue will result in a number. After performing some operations on it you will get a pair of numbers ( ; ) which are co-ordinates on the map. For example suppose your answer to a particular clue is 25 you may be given instructions such as (add 5 to your answer; divide your answer by 5) to give the co-ordinates (30; 5). Plot the point to find the location of your next clue. Some clues may give you both co-ordinates in one shot.

2. After you collect each clue, please gather in your respective rooms to collectively solve your problems. It is in your interest to cooperate. Use all the mathematics you have learnt thus far.

3. Each clue has a special *LETTER* (these are in a special font) - after you've picked up all the *LETTERS* unscramble them to find your treasure!

## JUNIOR TREASURE HUNT (Ages 11 to 13)

1.To find the place that I am hid, you'll have to find the next *B*id -

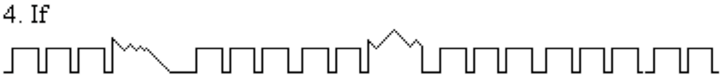
" 1...8...27...64.........

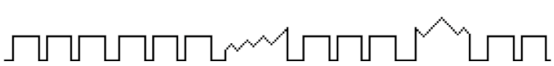
(subtract 23 from your answer; divide your answer by 5, then add 7)

2. One dark night I had to pick a pair of so*C*ks from my drawer. I have 5 red and 5 blue socks. How many do I have to pick out to have a pair? Get the answer and find me there! (multiply your answer by 8; add 1 to your answer, then multiply by 13)

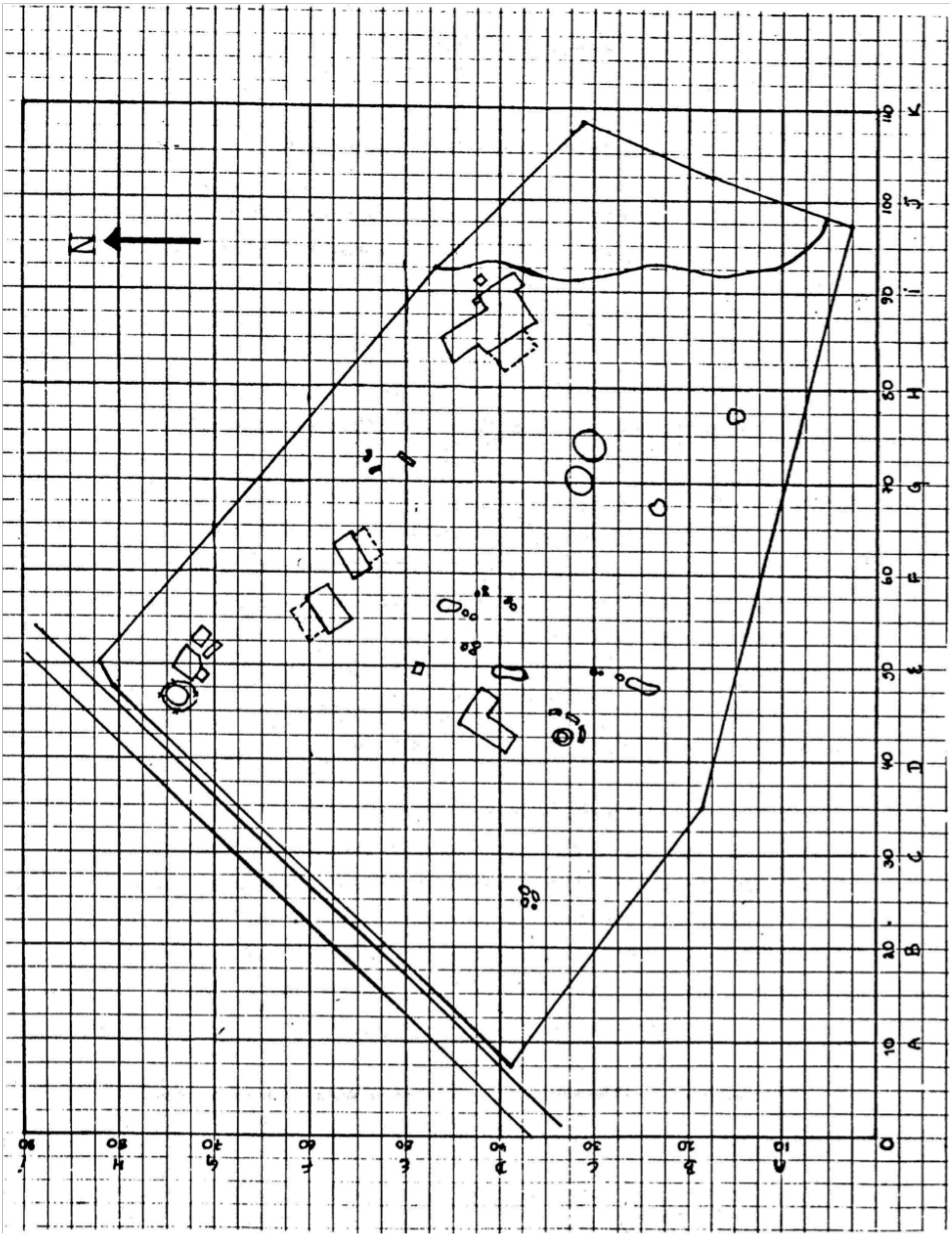
3. I had a w*E*ight of 13kg. It fell from my hand and split into 3 parts. With these three parts I can weigh all of 1 to 13 kgs. What are these 3 parts?
(multiply all the numbers in your answer, add 50; add all the numbers in your answer, add 2 )

4. If

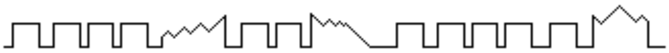


and 

**A map of Shibumi where the CFL Math Mela was held**



then what is



(multiply your answer by 10, then add 5; multiply your answer by 8, then add 2)

5. A teac*H*er once asked Gauss to add all the numbers from 1 to hundred;
Gauss thought for a minute, then said, "There's nothing in it! the answer I get is -"
(add all the digits in your answer, add 5, then multiply by three; add all the digits in your answer, then add 6)

6. There are four balls of equal size - three of them have the same weight and one of them is differe*N*t. What is the smallest number of weighings required to find the one which is different?
(multiply your answer by 25; multiply your answer by 41)

7. On your map, connect the points with co-ordinate*S* (35,1), (73.5,22.5) and (17.5,56.5). Find the point, which is equidistant from these three points.

**<u>SENIOR TREASURE HUNT (Ages 15-17)</u>**

1. A long long time ago, three people and a monkey were shipwrecked on an island. They spent the first day gathering coconuts, like good deserted folk. During the night, one man woke up and decided to take his share of the coconuts. He divided them into three groups. One coconut was left over so he gave it to the monkey. Then he hid his share and went back to sleep. Soon the second man woke up and did the same thing. After dividing the coconuts into three piles, one coconut was left over which he gave to the monkey. Then he hid his share, and went back to bed. The third man also did the same. The next morning after all the three woke up, they divided the remaining coconuts into three equal shares. This time no coconuts were left over for the monkey. What is the smallest number of coconuts they could have originally gathered?
(Multiply your answer by 3, then subtract 3; subtract 12 from your answer)

2. What is the value *O*f $\sqrt{7+\sqrt{48}}+\sqrt{7-\sqrt{48}}$ ?

(double your answer, then add 1; multiply your answer by 10, then subtract 2 )

3. "Every male bee has only one parent, a mother. Every female bee has two parents, a mother and a father. How many great-grand-parent*S* does a male bee have?

(add up all the digits of your answer, multiply by 4, then add 3; take away 50 from your x-co-ordinate)

4. On a true-false exam with ten questions, if a student answers all questions by guessing (and she has to get 8 out of 10 right to pass) what are her *C*hances of passing? Write your answer as a fraction.

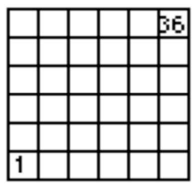
(subtract 3 from the numerator of your answer; subtract 2 from the numerator of your answer)

5. The area of a regular hexagon is 24 sq.cm. Interior to it, another regular hexagon is constructed by connecting the mid-points of the sides of the original hexagon.

What is the area of this newly constructed hexagon?

(multiply your answer by 5, then add 11; take away 5 from your answer )

6. How many shortest paths are there for a rook to move from 1 to 36?




(divide your answer by 4, then subtract 20; add 32 to your x-co-ordinate)

7. Draw a t*R*iangle connecting the following points - (57,78), (14,2), (99,2). Find it's centroid.

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# VII

## MATHEMATICS AND DRAMA

To the uninitiated this combination may seem strange! However during the Mathematics Mela we discovered that drama can be a powerful medium to teach the history of mathematics and also help children gain a personal relationship with numbers. Along with older students the 10 – 12 year olds participated in two skits/plays which were presented on the day of the mela. Students were involved in scripting and directing the play. The first was **‘Nirmala in Numberland: a play on numbers’**. The play, a take-off on ‘Alice in Wonderland’, is the story of a young girl in search of ‘infinity’. The drama plays on the everyday meanings of mathematical terms for numbers, like the ‘irrational number’. Although the purist may wince, this kind of association helps students remember these numbers and their properties more easily than the mere exposure to their definitions. The second was a play with a series of skits on the **History of Mathematics**. The skits traced the history of mathematics from ancient to modern times. In the years since the mela, whenever a mention of a mathematician is made, students immediately recall the associated skit – it seems to have taken root in their consciousness!

The next two sections are verbatim scripts that we used for the two plays. Teachers can either use the scripts as they are or adapt them to suit their purpose. It would also be interesting to write more plays, which explore drama as a pedagogical tool to teach the history of mathematics.

## NIRMALA IN NUMBERLAND

### A Play on numbers

*To see a World in a Grain of Sand*
*And a Heaven in a Wild Flower*
*Hold Infinity in the palm of your hand*
*And Eternity in an hour* - *William Blake*

Cast:
Nirmala
Older Sister
Negative Number (-1)
Two Complex Numbers (Mr. and Mrs. √-1)
Zero
Positive Numbers (1,2 and 3)
Irrational Number (√2)
Three Fractions: Vulgar, Improper and Proper
Transcendental Number π
Several Infinities: Aleph Null, etc.

Scene One

Two sisters, one with a book on Number Theory, the other lying with her head on her sister's lap, listening.

Older sister (reading from book): There are many kinds of numbers, like the natural numbers 1,2,3, 4 etc. But 0 is not a natural number. Then there are negative numbers, whole numbers and fractions, complex numbers like the square root of minus one, irrational numbers like the square root of 2 and transcendental numbers like π....

Nirmala: But do these numbers ever end? (Yawning)

OS: Well they don’t really, but if you go far enough you might get to infinity. You see (reads from book)...

N: Zzzzzzzz (fast asleep)

OS: Oh! What’s the use? Numbers are so fascinating for me, but they are just not real for you. (walks off with the book, still reading)

N: (talking in her sleep): Infinity, Infinity, where are you?

Enter the parade of numbers each miming some action. N wakes up to see negative number with his head down; the others have disappeared.

N: Good Morning!

-1: No! I don’t think it will last. True the sun is shining, but a huge plane may leave a trail of smoke.... and that’s the end of your good morning.

N: Who are you?

-1: No, you won’t know me. I am a negative number, minus one.  What about you?

N: Oh, I am Nirmala and I am looking for infinity.

-1: No, I suggest you don’t even try. You may get there and he may be out of town. In any case I don’t even know where you can find infinity.

N: Do you know any one else who might know?

-1: No.... well you can come home and ask my parents. They are Complex Numbers...
(walks) Are you coming?

N: Sure, thanks, I will.

Scene Two

-1: These are my parents Mr. and Mrs. √-1. You see they multiplied and I was born.

N: Hello! I am Nirmala and I am looking for infinity.

Mr. √-1: Hello! Please come in.

Mrs. √-1: Wipe your feet.

Mr. √-1: Sit down.

Mrs. √-1: Here's a chair.

Mr. √-1: Have some food.

Mrs. √-1: Have a bath.

Mr. √-1: It is so cloudy you must be feeling very hot.

Mrs. √-1: Have some water which is not cold.

Mr. √-1: You look like some one I don't know.

Mrs. √-1: Yes you remind me of yourself.

Mr. √-1: So, are you looking for infinity?

Mrs. √-1: have fun and the worst of Good Luck!

(These statements are made in quick succession)

N walks off while they are still talking.

-1 (calls out mournfully): Good Luck!

Scene Three

N hears a voice say “Hello”. Looks around and sees no one.

N: Who spoke just now?

Zero (hidden): When I am alone, I am as nothing. When I am with you I become something. Can you guess who I am?

N (thinking): Oh! You must be zero.

Z (appears and sticks to her): Absolutely!

N: Why are you sticking to me?

Z: I told you. If I don’t, I become nothing.

N: Okay then, why don’t you join me in my search for infinity?

Z: I will be right behind you.

Scene Four

A group of three numbers one, two and three are talking; moving hands and heads positively. N and Z approach.

Two: Number One, don’t be so odd.

One: Look at number three. He is odd too.

They turn to greet N and Z with a slap on the back.

Together:  Where are you off to on this beautiful day?

N: We are looking for infinity.
One: Oh! No problem.

Two: Success is guaranteed.

N: Thanks, we'll be off then.

Enter irrational number, √2.

√2: Wait! You can't ignore me.

One, two and three gesture to N to go on and whisper to her, "Don't listen to him, he is irrational".

N: Why? What do you mean?

Two: Look, I am a rational number. Let me explain. Eight by four is two, ten by five is two and eighteen by six is three. That is, we can all be expressed as ratios of two integers, but you can't say the same of him. That's why he is irrational.

√2: Even the Greeks were afraid of me! So you there, stop whispering.

One: Just watch his strange behavior.

√2 is throwing salt over his shoulder, fingers are doubly crossed. One, two and three walk off.

N: what are you doing?

√2: Some one complimented me today- so now obviously something terrible is going to happen to me. That's why I am throwing salt over my shoulder.

N: How will that help?

√2: Well- it just does- like keeping my fingers crossed.

N (sniffing): Is something burning?

$\sqrt{2}$: Yes! That's the toast, if you burn the morning toast you are sure to have a good day.

N: But ... but ... how does that follow?

$\sqrt{2}$ (thinks and seems confused): stop asking so many questions!

N: Could you help us search for infinity?

$\sqrt{2}$: I don't know how you'll find it. But I can tell you one thing. Don't start on a Saturday.

Scene Five

Big Medical Plus sign seen. Three fractions (vulgar, improper and proper) are seated in slings and casts. As N and Z approach the vulgar fraction whistles and the proper fraction pulls him back.

PF: Shhh... Behave yourself! What will all these people say?

N: What's happened? (Improper fraction clears his throat and spits. Proper fraction is scandalized again and tries to stop him)

Z: Oh! they are fractions, not whole numbers. This is a vulgar fraction, this is an improper fraction and this is a proper fraction. (They make appropriate gestures)

N: will the doctor be able to help them?

Z: Yes I am sure he can, he will probably prescribe reciprocals.

Scene Six

They pass on and hear sounds of chanting Om! Om! A transcendental number $\pi$ is in meditation.

N: That must be it! I am sure that's infinity. Are you, sir?

Pi (slowly opens eyes): No! I am a transcendental number. I am also irrational, but somehow beyond irrationality. You're looking for infinity? I'll give you a hint. If you count my digits you may find infinity....

N tries to counts his fingers and toes, but Z impatiently pulls him away.

Z: Not that Nirmala, that's not what he meant but... wait ....can it be... ? Look over there....

Long pause, when slowly turns and approaches a long line of infinities extending all around the audience etc.

N (asks the first one): Are you infinity?

" Yes... yes...yes...yes..." echoes around the place.

N: How can there be so many of you?

Aleph Null: There are infinitely many infinities! Who were you searching for? Me?

" Or me?.... Or me?.... Or me?.... Or me?.... Or me?...." The infinities crowd around her like a pack of cards and she falls to the ground. They disperse, she wakes up, yawns, looks around, older sister comes in.

OS: Sleepyhead!

N: You wouldn't believe what an adventure I just had.

OS: Right, of course, you were fast asleep! Shall we continue the lesson then?

N (thinks): Yes, let's. Maybe I can teach you a few things about numbers!

THE END

## THE HISTORY OF MATHEMATICS: A BRIEF TOUR THROUGH COMMENTARY AND SKITS

*Mathematics is a unique aspect of human thought, and its history differs in essence from all other histories.* - Isaac Asimov

Mankind has been using mathematics for 50,000 years. Even before he needed to count his sheep, cave man had to keep track of his wives.

Enter: A cave man with many girls behind him, grunting “one, two, many...”.

Man also knew primitive geometric shapes. As time went on these became more sophisticated and abstract. It is believed that the Mesopotamians, the Chinese and the Hindus knew the Pythagoras Theorem before Pythagoras himself. The Mesopotamians (~ 2400 BC) had a complete table for the Pythagorean triplets.

Enter: three students, representing triplets, showing the equation
$3^2 + 4^2 = 5^2$.

They also invented the sexagesimal system which is base 60; a system still used today whenever one looks at a watch. Geometry really took off with the Greeks and even today their influence is felt: geometry was tied up with their worldview.

Enter: Plato saying, “ No one who knows not geometry shall enter these portals. What are you people doing here anyway?”

Number worship by the Pythagoreans was carried to extremes. They believed that each number had an attribute. For example, number 2 was female, number 3 was male and number 5 was marriage.

Enter: A couple showing 3 + 2 =5.

Now, who was Pythagoras? He was one of the most well known mathematicians from ancient Greece. He was a cult figure, who had gathered around him a group called the Pythagoreans. They believed in the mystical powers of numbers. Pythagoras (~ 540 BC) was a vegetarian, and is believed to have come to India, and taken back with him the idea of transmigration of souls. During his time, irrational numbers were discovered.

Enter: Pythagoras, Hippasus, and other Pythagoreans. They sit around chanting and worshipping numbers, suddenly Hippasus jumps up in great agitation and says, “ Brothers! I have made a discovery, I have found a number that can not be expressed as a ratio of two numbers!” The group is very disturbed, but Pythagoras, calms them down and says “ Brother Hippasus, it is not possible. Will you explain yourself?” Hippasus, nervous, goes up to a

black board and says, "I have a proof. Using your own Theorem, I calculated the diagonal of a unit square to be $\sqrt{2}$ . He then goes to demonstrate that $\sqrt{2}$ is irrational.

One story has it that the Pythagoreans drowned Hippasus for his heretical discovery.

Three hundred years later after Pythagoras, another Greek, Euclid (~ 300 BC) wrote his best seller, 'Elements'. Fortunately, it was not the time of 'publish or perish', as there was no publisher. Centuries later, Abraham Lincoln worked out a few of Euclid's theorems every day to keep his mind sharp.

In the same century as Euclid, lived a man of extraordinary talents.

Enter: Archimedes's wife, and sits in the centre. She says to the audience, "This husband of mine, Archimedes! Always having a bath, wasting so much bath oil!" Just then Archimedes enters, his body bathed in oil and dressed in a towel yelling "Eureka! Eureka!" His wife looks up in surprise. Archimedes dances around her a few times saying, "I have conquered π! I have conquered π!" and dances out again. She looks after him dazed, and says, "I never knew you could cook...?"

Although we can never think of Archimedes (~ 212 BC) without "Eureka" and the bath story, it is a lesser-known fact that he won a battle single handed against the Romans by inventing war machines like the catapult, and by burning sails of the enemy ships with the help of huge mirrors. He is reputed to have said, "Give me a place to stand on, and I will move the world."

While the Greeks concentrated on Geometry, the Indian Mathematicians Bhaskara and Aryabhata (476- 550 AD) were interested in algebra and number theory. Aryabhata wrote the " Aryabhatiya", akin to Euclid's "Elements", one of the oldest mathematical texts known, dating back to the Roman Empire. Aryabhata computed π up to four decimal places.

Long ago, three brothers were left thirty-five camels by their father. According to his wish, half of them were to go to the eldest son, one-third to the second and one-ninth to the third. He had also said that none of the camels were to be killed. How, then did the brothers receive their inheritance?

Enter: An Arab and his friend pulling a camel behind them. The Arab turns to his friend and says, "It is very simple to make the division fairly. But let me add to the inheritance of 35 camels this splendid beast of yours, which makes 36 camels in all. The oldest brother would have received half of 35, that is 17.5 camels. Now he will receive half of 36, that is 18. The middle brother would have received one third of 35, which is 11.67 camels. Now he will receive 12. The third brother would have received one ninth of 35, which is 3.89, and now he will receive 4. Each of them has received more than his fair share. By this advantageous

division, 18 belong to the oldest, 12 to the middle and 4 to the youngest, adding up to 34. Two remain, one belongs to you, and the other I will keep as payment".

That could well have been Abu Jafar Mohammed ibn Moosa al-Khwarizmi (830 AD), which means Mohammed, the father of Jafar and the son of Moosa, from Khwarizmi. He wrote a famous book, 'Hisab al Jabr w`al Muqabala'. The word algebra comes from the title 'al-Jabr', and it is to some extent the complicated nature of laws governing inheritance that encouraged the study of algebra in Arabia.

Apart from being mathematicians, the Arabs were preservers of mathematical knowledge. The Greek texts were lost to the western world and re-discovered in Sanskrit, Pehlavi, and Syriac by the Arabs, and translated back into Greek and Arabic. The number system, which we used today, was invented by the Hindus, but the Arabs spread it in the Western world. It is therefore called the Hindu-Arabic numeral system.

In the 1100's India produced a famous mathematician and astronomer, Bhaskaracharya (1114-1185 AD). One of the books he wrote was "Leelavati". There is a charming story about how this book came to be written.

Enter: Bhaskaracharya and Leelavati enacting the story as it is narrated.

Leelavati was the name of Bhaskaracharya's daughter. From her horoscope he had discovered that the auspicious time for her wedding would be a particular hour of a certain day. He placed a cup with a small hole at the bottom of a vessel filled with water arranged in such a way that the cup would sink at the beginning of the propitious hour. Everything was ready, Leelavathi, out of curiosity bent over the vessel, and a pearl from her dress fell into the cup and blocked the hole. The lucky hour passed and the cup didn't sink. To console the dejected daughter, who would now not get married, Bhaskaracharya wrote her a manual of mathematics.

Bhaskaracharya points to some swans on a lake, and asks Leelavati, "Beloved daughter, out of a group of swans, 7/2 times the square root of the number are playing on the shore of the lake. Two remaining swans are fighting in the water. What pray is the total number of swans?". Leelavati ponders for a while then answers "Father, there are 16 swans".

Bhaskaracharya apparently solved an equation, which was set forth as a teaser by the French mathematician Fermat 500 years later. Fermat (1601-1665) was a lawyer who did

mathematics in his spare time. What has become famous as Fermat's last theorem was finally solved in 1995 by Andrew Wiles, a British mathematician working at Princeton University

Enter: Fermat, and sits in his study, working. Suddenly, Wiles appears from the future! He is very awed, and says, "Are you Monsieur Fermat? I am Andrew Wiles, from the 20 century. I am honoured to meet you, and I'm happy to say that I've proved it!" Fermat looks bemused and asks, "Proved what?". Wiles says, "The problem of $x^n + y^n = z^n$, of course! Hundreds of mathematicians have been trying to prove, for hundreds of years, your claim that there are no integer solutions to this equation for n larger than $2^!$ I finally solved it. But I had to use the Tanayama-Shimura conjecture, and my proof was way too long to have fit in your margin...it was a thousand pages, and -". Fermat interrupts, "A thousand pages? But my dear fellow, how unnecessary! My proof is very simple, beautifully short, here let me show you -". Wiles comes forward excitedly, but suddenly Fermat checks his watch and says, "Ooops, very sorry, I have to run, there isn't enough time to explain it right now, got to go,..." and he dashes off stage leaving Wiles gaping after him.

Mathematics took a great leap forward during the time of Descartes (1596-1650), the French philosopher-mathematician who made a link between algebra and geometry. As a boy, he was sickly, and was allowed by the indulgent father and teachers to lie in bed the whole morning. He used his time to think about mathematical problems. In the winter of 1649, when he was 53 years old, he went to Stockholm to become the private instructor to Queen Christina of Sweden.

Enter: Descartes, sleeping on his bed. He wakes suddenly, saying "Oh! I'll be late! The Queen's class!" He rushed to the castle. Queen Christina is impatiently awaiting him, and says "Hurry up, teach me something, I have to go riding after this." Descartes begins to teach her about the coordinate axes, X and Y. He tells her he has invented this good way to help map her royal grounds and locate any position quickly and easily. When the lesson is over, she leaves in a great hurry. Descartes looks after her tiredly and says, "I think, therefore I am. She coordinates, therefore she is!"

Now we come to Galois (1812-1832). Young geniuses whose lives were cut short in their youth are a part of the tradition of the Romantic Age. Dogged by ill-luck and misfortune, his mathematical abilities were not recognized till after his death at the age of 20 in a duel. What is currently known as Galois theory has very wide ramifications. His work has been used to show that the three great problems of antiquity-squaring a circle, doubling a cube and trisecting an angle are impossible to solve using just a straight edge and compasses.

With the invention of Calculus by Newton (1642-1727) and Leibnitz (1646-1716), mathematics flourished in the Western world. In India, however, mathematical activity slowed down, until in January 1913, G.H. Hardy, a famous Cambridge mathematician, received a sheaf of pares filled with theorems from far-away Madras. Hardy later said, " A single look at them is enough to show that they could only be written by a mathematician of

the highest class". The author of these theorems was, of course, Srinivasa Ramanujan (1887-1920). At Hardy's invitation, he worked at Cambridge for six years. Ramanujan was deeply religious, mathematically brilliant, and in many ways extremely child-like. A good friend of his at Cambridge was P.C. Mahalanobis, the Bengali statistician, who later founded the Indian Statistical Institute.

Enter: Ramanujan, making hot rasam and rice in his kitchen. His good friend Mahalanobis comes to visit him for lunch. "Delicious smell!", he says. "Have you seen this puzzle in the latest issue of the Strand? It says, a house is on a long street, numbered on this side one, two, three and so on, and that all the numbers on one side of him added up exactly the same as all the numbers on the other side of him. The number of houses is more than fifty but less than 500. What is the number of the house? I think I have a possible solution! It is 288!" Ramanujan says, while stirring the rasam, "Actually, there is a general solution to this. Take it down please. The answer is a continued fraction......" As he reels off the answer, his friend writes it down on a blackboard in a dazed way. At the end he says, "You are a genius, my friend!"

Ramanujan's answer turned out to be correct and completely unexpected. But his genius in mathematical matters was not to be seen in his daily living!

Ramanujan is lying in bed, shivering from cold. Mahalanobis comes in and says, "How are you? Are you well, warm enough?" To which Ramanujan answers, "Well, my friend, except it is very very cold, and there is nothing to cover myself with". Mahalanobis goes up to him, and points to the bedclothes on which he is lying. "But these are blankets, Ramanujan! They are meant to cover yourself with!" Ramanujan exclaims "Oh! I thought these were to sleep on, and not under...you are a genius, my friend!"

Ramanujan's insight seems to have come from the gods themselves and he was perhaps unconcerned with the logical foundations of mathematics. His western counterparts, however were attempting to derive mathematics purely from logic, to demonstrate that mathematics, as a formal system, was free of logical contradiction. To their dismay, an Austrian logician and mathematician, Kurt Gödel (1906 -1978) published a landmark paper in 1932, proving that any formal mathematical system cannot be both consistent and complete at the same time. This shook many mathematicians, forcing them to acknowledge that there will remain certain results in mathematics that are undecidable. Gödel's good friend at Princeton was Albert Einstein, who was very protective of this fragile and eccentric man.

Enter: Gödel, in his house. Einstein knocks on his door, and says, "Hello, Gödel! Shall we go for a walk?" Gödel replies, "If it is Tuesday, we shall walk. It is Tuesday. Therefore we shall walk." They both leave the house on their walk. Gödel tells Einstein about the problem he is working on, and says, "You see, I have discovered something, which will shock the world of mathematics! I have found that no axiomatic system can be both complete and consistent at the same time. I did it by inventing a statement in mathematics whose truth cannot be proven." Einstein is curious and says, "What do you mean, is it neither true nor false?" Gödel says, "Consider the statement 'THIS STATEMENT IS FALSE'. If it is true,

then it is false. But then it can’t be true. So it is false. But if it is false, then it must be true. And so on!” They walk off

Gödel’s work has left an indelible mark on mathematics and the philosophy of mathematics. Mathematics in the 20th century has grown exponentially and affects all aspects of modern life. It will perhaps continue to be a very important aspect of human knowledge for some time to come. However most of present day mathematics is inaccessible to the lay man and most people feel alienated from mathematics. We hope that we have been able to share the human side to mathematics and also been able to convey the beauty and charm of the subject.

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[MK] Morris Kline, *Mathematical thought from ancient to modern times*, **Vol. 1,2,3**, Oxford University Press, Oxford, 1972.

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## VIII

## ON ORGANIZING A MATHEMATICS MELA

“ *The average student needs emotional and intellectual satisfaction now, not just in five or ten years’ time, when they become adults! The pleasure and satisfaction that children find in drawing, in colour, and in modeling can, perhaps, contribute to their learning of mathematics. And the development of geometrical concepts and skills can be applied to help the child or the student gain pleasure from creative art. If we can achieve this,*

**Plate VIII.1. A tessellation mural by students of CFL.**

*the child will appreciate the relevance and the power of mathematics in a personal way, and will associate the pleasure of creative art with that of creative mathematics.”*

- Julian Williams

### Why A Mathematics Mela?

We feel that a Mathematics Mela can help change the attitude children have towards mathematics. For example, their belief that the understanding of mathematical concepts comes quickly or not at all. It is a great chance to teach children topics not usually covered in the regular syllabus and to do projects and investigations in mathematics which usually take a relatively long span of time. These can be used to demonstrate concretely that mathematics is beautiful. It gives children an opportunity to work on mathematics together and co-operatively rather than as individuals. Melas create a festive atmosphere in the school and bring about a sense of togetherness.

**The Mathematics Mela at CFL**

Students worked on a project for 2-3 hours a week for a whole term. Topics we found suitable were:

| | |
|---|---|
| Tessellations | appropriate for age 5-12 yrs |
| Platonic solids | appropriate for age 13-15 |
| Project on $\pi$ | appropriate for age 15-18 |
| Fractals | appropriate for age 15-18 |
| Probability and Statistics | appropriate for age 15-18 |
| Recreational Mathematics | appropriate for all ages |

Apart from these we had activities involving the whole school. We had talks and presentations by experts from outside the school. We created a 'math softboard', where cartoons, articles and puzzles were regularly put up. We conducted whole school assemblies where groups made presentations, puzzles appropriate to each age group were posed and solutions to puzzles were discussed. Children put up plays with mathematical themes. We had a treasure hunt based on mathematical clues.

Children, teachers and even some parents painted/stitched their favourite fractal, tessellation, or any other mathematical idea onto a piece of clothing. On the day of the mela they wore their math mela clothes!

CFL Mathematics Mela was held on March 23rd, 1996. We had a booth or a stall for each project managed by a group of children and perhaps a teacher or two. The main emphasis was to make the sites as interactive as possible. Children explained their work and also the mathematics involved. The challenge of having to explain mathematics to a lay audience was an excellent exercise in itself and also helped reinforce their understanding of the underlying concepts. Wherever possible we conducted a small workshop on the topic concerned, for example teaching how to tessellate.

We had a *Santhe* (a village fair or a market day usually held in the central part of the village, very colourful and generally a hub of activity) in the central courtyard of the school. Children of all ages had stalls with games and puzzles. People wandered from stall to stall solving puzzles or playing games. Children carried puzzles in trays 'hawking their ware' and challenging people to a nice puzzle. The day ended with two plays and a dance.

**Organising a Mela in your School**

The school as a whole should be willing to commit the time and resources necessary for a mela. For the mela to be successful the time required would be 2-3 hours a week for a term with intensive work about two weeks before the day of the mela.

Organize children into groups with a teacher coordinating each project. As mentioned above the topics we found suitable were: Tessellations, Platonic Solids, Fractals, a project on $\pi$, Probability and Statistics. Other topics that come to mind are exploring patterns in numbers, making abacuses and colourful Inca *quipus,* networks and graph theory, Fibonacci sequences and Pascal's triangles, symmetry, paper folding.

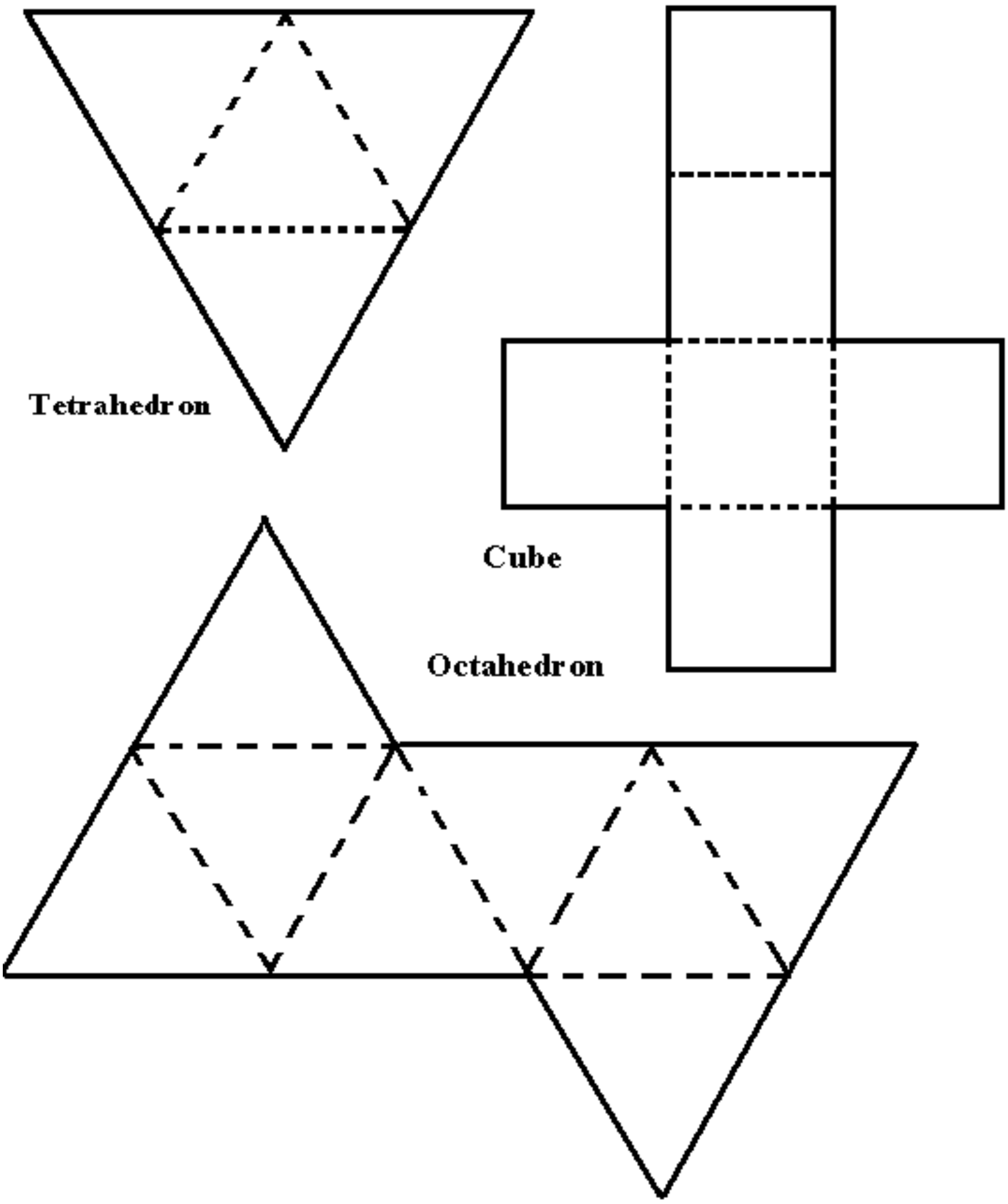
The teacher conducting the project should have a clear outline on how it will evolve, but should be flexible enough to allow for spontaneous ideas and creativity. The teacher should have all the required material ready. It is a good idea to document the project as it is happening and also to create worksheets to teach the mathematical concepts involved.

Organize whole school activities like a math softboard, assemblies, problem solving sessions, treasure hunts and plays.

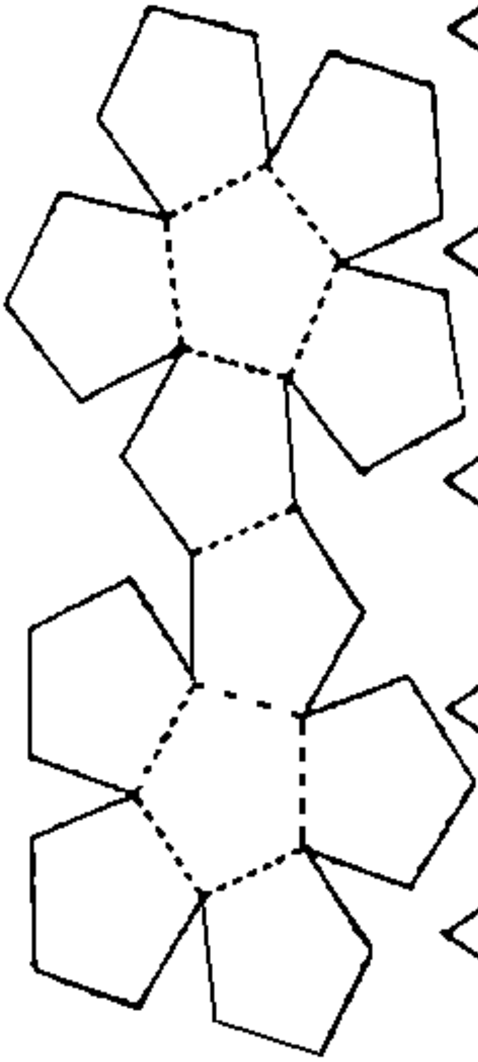
For the day of the mela organize booths to present projects, and make the interaction with the audience participatory.  Conduct mini-workshops where possible, and have a Santhe for the presentation of puzzles and games. Plays help gather the audience.

# Appendix 1

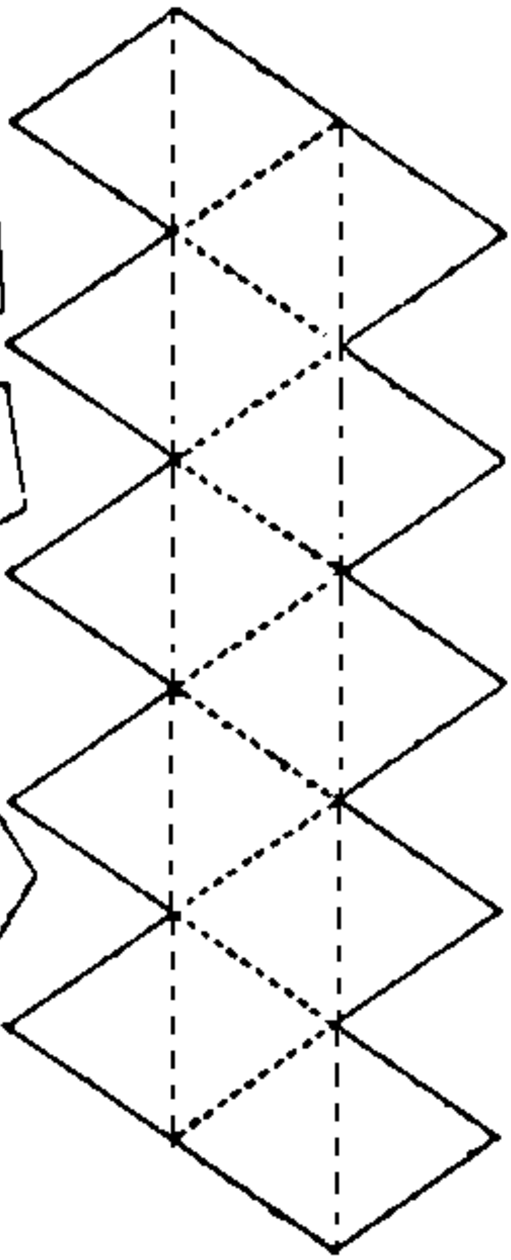
## Nets to create Platonic Solids



# Nets to create Platonic Solids



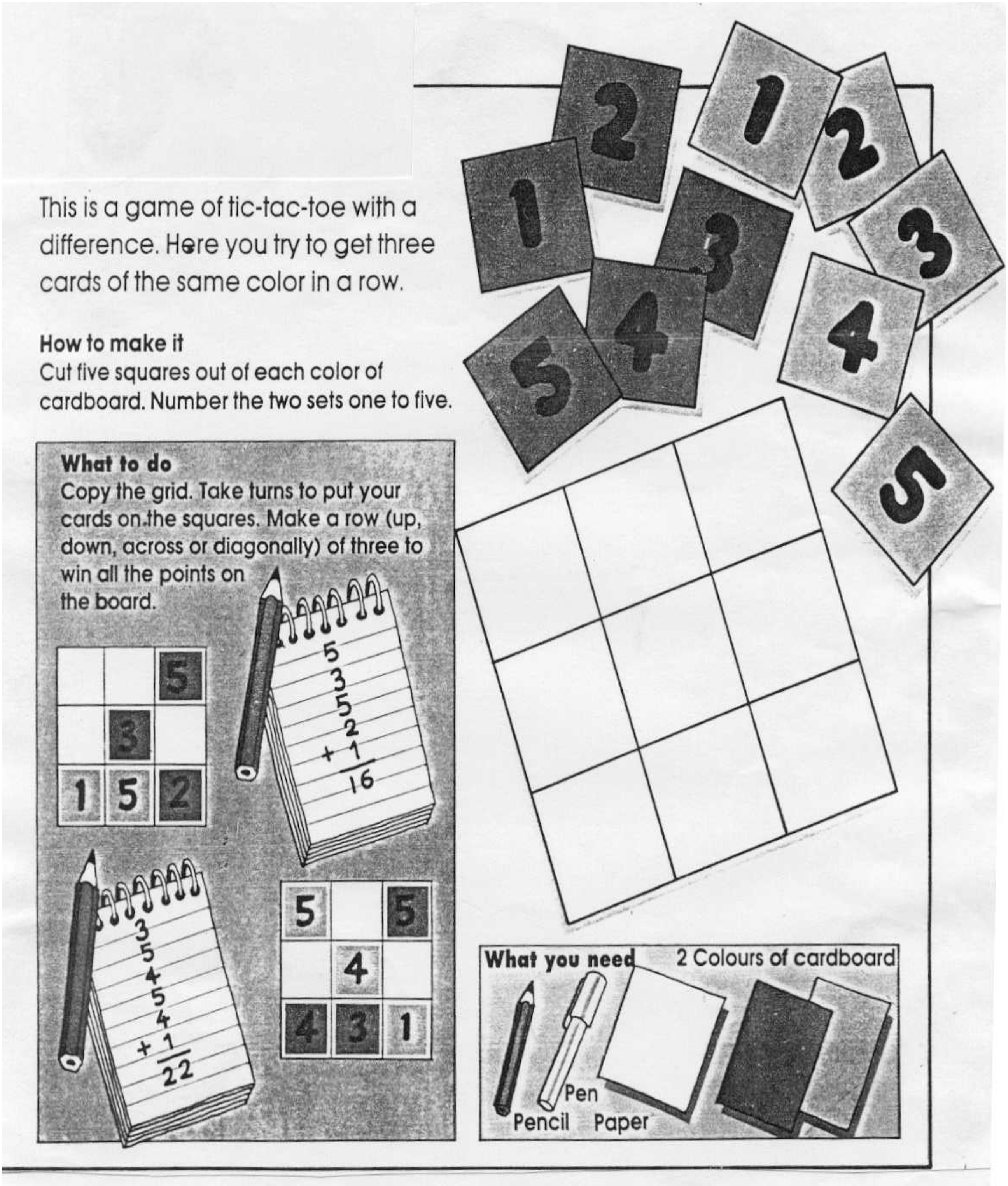
**Dodecahedron**



**Icosahedron**

## APPENDIX 2A

## Tic-Tac-Toe



This is a game of tic-tac-toe with a difference. Here you try to get three cards of the same color in a row.

**How to make it**

Cut five squares out of each color of cardboard. Number the two sets one to five.

**What to do**

Copy the grid. Take turns to put your cards on the squares. Make a row (up, down, across or diagonally) of three to win all the points on the board.

**What you need** 2 Colours of cardboard

## Appendix 2B

### Collapsing Sphere

**What you need**

Card paper of three different colours/patterns, coloured on both sides.

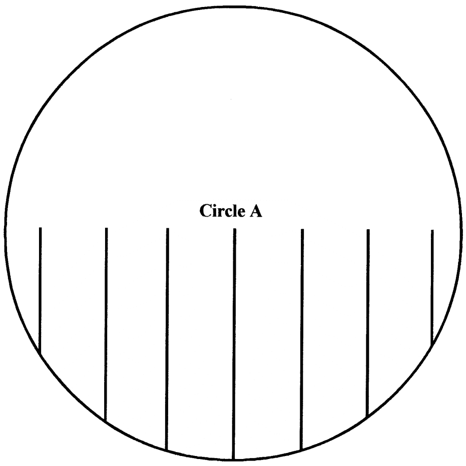
A pair of compasses.

A pair of scissors or cutter.

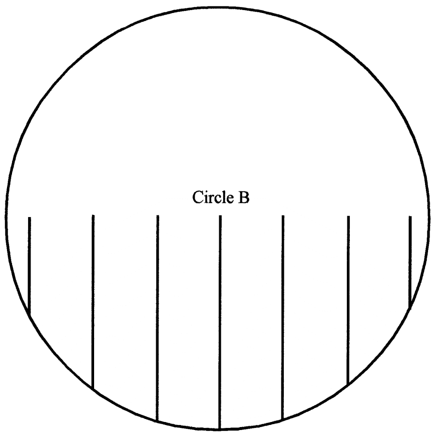
The four circle stencils shown below.

**Procedure**

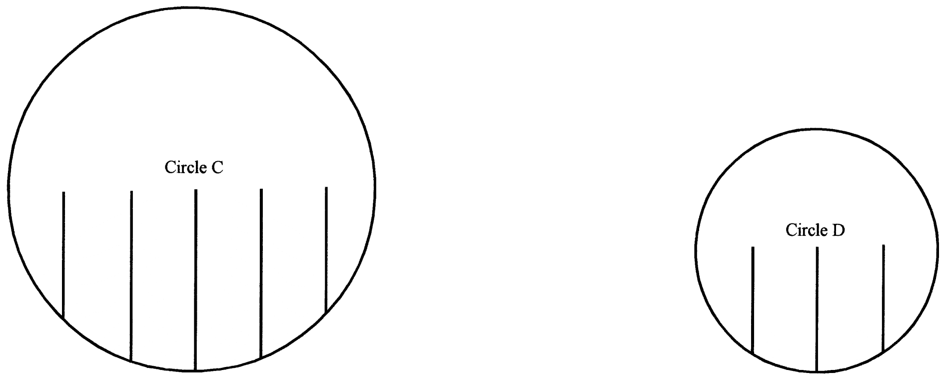
Make two circles using the A stencil and four using the D stencil from one colour card. Then Make four circles using the stencil B and four using the C stencil from the other two cards. Draw vertical lines as shown on all 14 circles, and cut along these lines.




Attach the two large A circles at right angles to each other, in the centre.




On either side of the vertical A circle, fit in two C circles and finally two small D circles. On either side of the horizontal A circle, fit in two B circles, followed by two C circles and finally two small D circles.

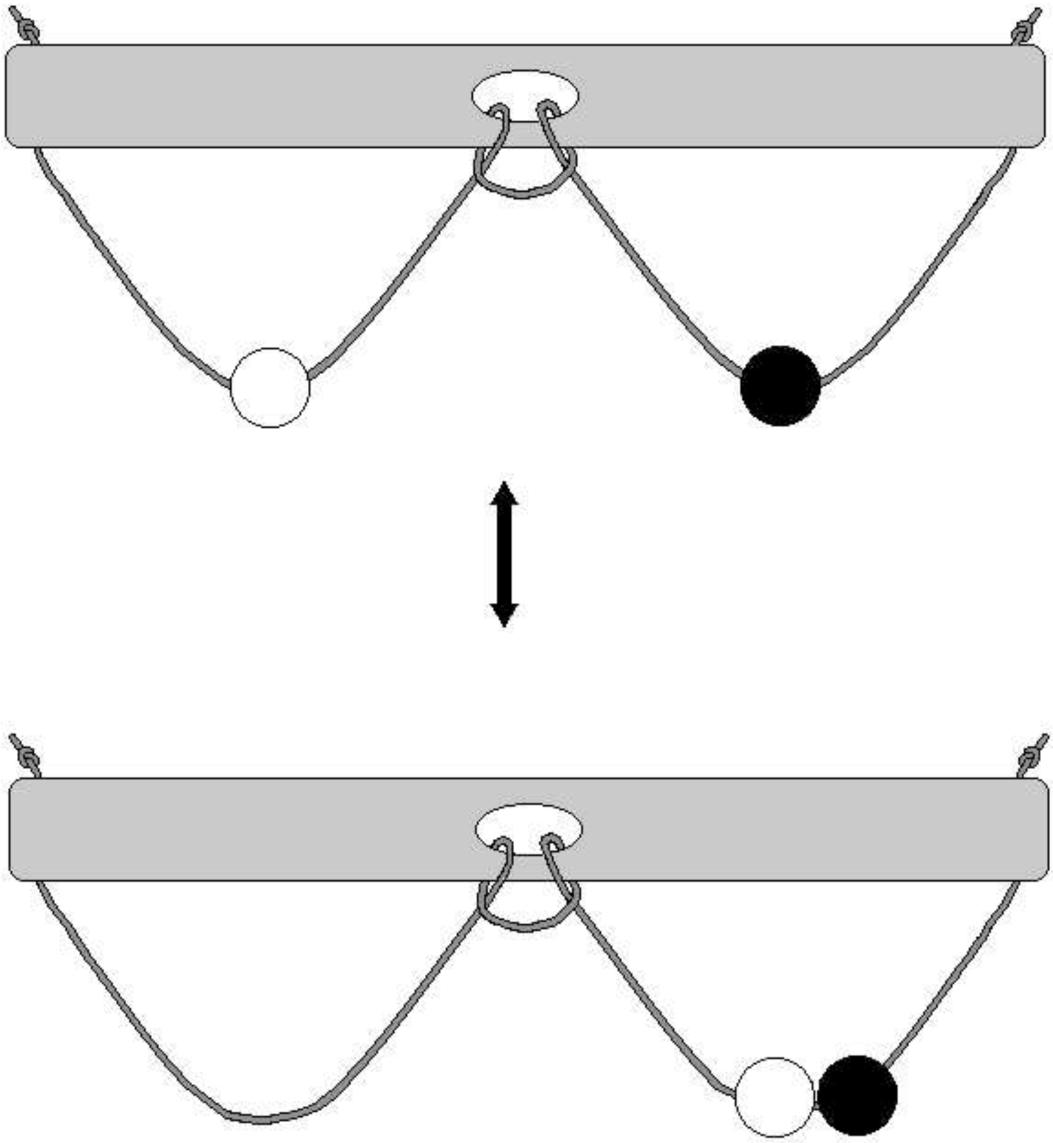



Now the sphere is ready.  It can be collapsed by pressing on one side – the circles will compress and lie flat.

## Appendix 2C

## A Topological Puzzle

Try to get a bead across the loop without using a pair of scissors !



## Appendix 2D

## SAMPLE FOLK PUZZLES[♣]

**Broken Eggs**

An egg trader was moving along a road selling eggs. An idler who didn't have much work to do started to get the egg trader into a wordy duel. This grew into a fight and he pulled the basket with eggs and dashed it on the floor. The eggs broke. The trader requested the Panchayat (five member committee) meeting to settle the dispute. The Panchayat asked the trader how many eggs were broken. He gave the following response:

If counted in pairs, one will remain;
If counted in threes, two will remain;
If counted in fours, three will remain;
If counted in fives, four will remain;
If counted in sevens, nothing will remain;

**Answer:** 119 eggs

**A Flock of Sparrows**

A sparrow was sitting on a branch of a tree. At that time a flock of sparrows was flying above the tree. The sparrow that was sitting called out to the flock of sparrows: "Oh! One hundred sparrows! Where are you going? Come and sit on this tree and take rest and then continue your journey". Hearing this, one of the sparrows from the flock said, " We are not hundred. We, a similar flock like us, one half of that, one half of that and you together will make one hundred". If that is so, how many sparrows were there in the flock that was flying?

**Answer:** 36 sparrows

**Oil Merchant**

In a village, there lived an oil merchant, who used to crush oils seeds, extract oil and sell it. After several days of hard work, he got ready to take the extracted oil to the market. On his way to the market he came across a Vinayaka temple. He entered the temple and prayed,

[♣] Adapted with permission from A.Rampal, R. Ramanujam and L.S. Saraswati, *Numeracy counts!* National Literacy Resource Centre, Mussoorie, 1998, pages 93-96.

"Today, if my business goes well, on my way back, I will light the lamps in the temple using one measure of oil."

He moved on. As he crossed some more distance, he saw a temple of the Goddess. He went in and again prayed the same way he prayed to Vinayaka. He continued his journey and reached the Ayyanar temple at the village limits. He again prayed the same way.
He went to the market. He had good business that day. In the evening as he was returning, he took sufficient quantity of oil in a pot to fulfil his vows in the three temples. He reached the Ayyanar temple. He saw a small tank of water near the temple. He kept the pot on the tank-bund and he got into the tank in order to wash his feet, hands and face. At that time a crow came and sat on his pot and tilted it. The oil flowed out on the ground. The merchant came running and straightened the pot. A small quantity of oil remained in the pot. A good quantity of oil was spilt on the ground.

With the quantity of oil left in the pot, the merchant went to the Ayyanar temple. He expressed his feelings of not being able to fulfil his promise. Ayyanar sympathised with him and blessed him saying, "May the quantity of oil in the pot be doubled." And so it happened. The oil trader fulfilled his vow of lighting lamps in the temple using one measure of oil.
He covered some distance. He reached the temple of the Goddess with the pot containing a little quantity of oil. Here again he narrated the happenings and expressed that he felt sorry that he was unable to do what he had vowed. The Goddess blessed him by doubling the quantity of oil in the pot. The merchant fulfilled his vow of lighting lamps in the temple using one measure of oil. A little quantity of oil was left in the pot.

The merchant walked further and reached the Vinayaka temple. With all devotion he narrated all that happened and expressed with sadness his inability to fulfil his vow. Vinayaka also blessed him with double the quantity of oil. The merchant took out one measure and lighted the lamps. The pot was empty now.

What was the quantity of oil in the pot when the merchant straightened the pot after driving the crow away?

**Answer:** 7/8 measure

**Milk Vending**

A milk vendor had several customers to whom he was selling milk every day. He generally brought milk in big metal pots and measured with traditional standard volume measure and supplied any quantity between one and eight measures. One day he forgot to bring in his standard volume measure. The customers also didn't have any clean volume measure for him to use. The capacities of the two metal pots he had were in terms of three measures and five measures. He used these two metal pots and measured milk from one to eight measures. How did he do it?

**Answer:**

| | |
|---|---|
| 1 measure = (2x3) – 5 | 5 measures = 5 |
| 2 measures = 5 – 3 | 6 measures = 2x3 |
| 3 measures = 3 | 7 measures = 5 + (5-3) |
| 4 measures = 2(5 - 3) | 8 measures = 5 + 3 |

**Broken weight measure**

A 40 palams (one viss) weight measure dropped down. It broke into four pieces. With these four pieces it was possible to weigh things from 1 palam to 40 palams. How much was the weight of each of the broken pieces?

**Answer:** 1,3,9,27 palams

## CREDITS

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Plate IV.3 – photographed by students of CFL.

**Chapter I**
Figure I.5. a. A tessellation of hexagons, squares and triangles, coded (6,4,3,4), from the Shibam-Kawkaban, a minaret in Yemen; Figure I.5. b. A tessellation of hexagons, dodecagons and squares, coded (6,12,4), from the Shibam-Kawkaban, a minaret in Yemen.
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